Domains and Games

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Generalised domain theories: stable domain theory, bidomains (Berry); sequential algorithms (Berry, Curien); game semantics (AJM, HO); domains as presheaf categories ($e.g.$ Girard's quantitative domains); categorical axiomatisations; ...

arose in answer to limitations of traditional domain theory:
operational semantics; nondeterministic dataflow; probability and higher types; probability and nondeterminism; concurrency; ...

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Event structures and their maps

An **event structure** comprises $(E, \leq, \mathrm{Con})$, *events* E, a partial order of *causal dependency* $\leq$, and *consistency* a family Con of finite subsets of E, s.t. $\{e' \mid e' \leq e\}$ is finite, ...

Its **configurations** $\mathcal{C}^\infty(E)$ comprise those subsets $x \subseteq E$ which are

consistent, *i.e.* $X \subseteq_{\mathrm{fin}} x \Rightarrow X \in \mathrm{Con}$, and

$\leq$*-down-closed*, *i.e.* $e' \leq e \in x \Rightarrow e' \in x$.

$(\mathcal{C}^\infty(E), \subseteq)$ *is a dI-domain (Berry) and all such are so obtained.*
Often concentrate on the **finite configurations** $\mathcal{C}(E)$.

A **map** of event structures $f : E \to E'$ is a partial function $f : E \rightharpoonup E'$ such that, for all $x \in \mathcal{C}(E)$,

$$fx \in \mathcal{C}(E') \text{ and } e_1, e_2 \in x \ \&\ f(e_1) = f(e_2) \Rightarrow e_1 = e_2\,.$$

Maps *reflect causal dependency locally:* $e', e \in x \ \&\ f(e') \leq f(e) \Rightarrow e' \leq e\,.$

Concurrent games

Games and strategies are represented by **event structures with polarity**, an event structure $(E, \leq, \mathrm{Con})$ where events E carry a polarity $+/-$ (Player/Opponent), respected by maps.

(Simple) Parallel composition: $A \| B$, by juxtaposition.

Dual, $B^{\perp}$, of an event structure with polarity B is a copy of the event structure B with a reversal of polarities; this switches the roles of Player and Opponent.

Concurrent plays and strategies

A **nondeterministic play** in a game A is represented by a total map

$$\begin{tikzcd} S \arrow[d, "\sigma"] \\ A \end{tikzcd}$$

preserving polarity; S is the event structure with polarity describing the moves played.

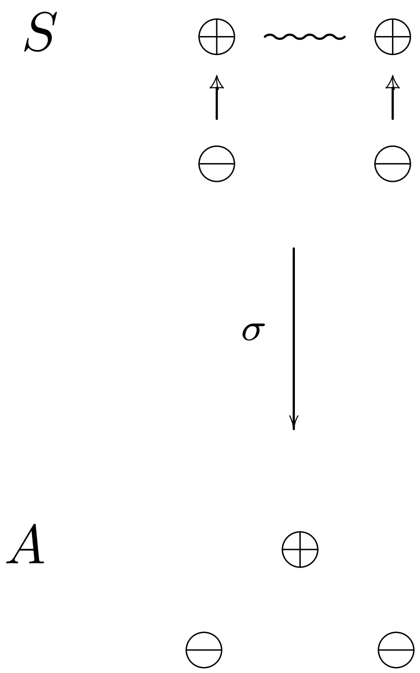
A **strategy in** a game A is a (**special**) nondeterministic play $\sigma : S \to A$.

A **strategy from** A **to** B is a strategy in $A^\perp \parallel B$, so $\sigma : S \to A^\perp \parallel B$.

[Conway, Joyal]

NB: A strategy in a game A is a strategy for Player;
a strategy for Opponent - a counter-strategy - is a strategy in $A^\perp$.

A strategy - an example



configurations of S = *"states of play"*

configurations of A = *"positions of the game"*

The strategy: answer either move of Opponent by the Player move.

Example: copycat strategy from A to A

$$\mathbb{C}\mathbb{C}_A$$

$$\begin{tikzcd}
A^{\perp} & & & A \\
\overline{a}_2 & \ominus \arrow[r, dashed] & \oplus & a_2 \\
\overline{a}_1 & \oplus \arrow[u] & \ominus \arrow[l, dashed] \arrow[u] & a_1
\end{tikzcd}$$

Composition of $\sigma : S \to A^{\perp} \| B$, $\tau : T \to B^{\perp} \| C$ via pullback:

Ignoring polarities, the composite partial map

$$
\begin{tikzcd}
 & T \circledast S \arrow[dl] \arrow[dr] \arrow[dd, dotted, tail, "\tau \circledast \sigma"] \arrow[rr, dashed] & & T \odot S \arrow[dd, dashed, "\tau \odot \sigma"] \\
S \| C \arrow[dr, "\sigma \| C"'] & & A \| T \arrow[dl] & \\
 & A \| B \| C \arrow[rr, "A \| \tau"] & & A \| C
\end{tikzcd}
$$

has partial-total factorization whose defined part yields

$$T \odot S \xrightarrow{\tau \odot \sigma} A^{\perp} \| C$$

on re-instating polarities.

For copycat to be identity w.r.t. composition

a **strategy** in a game A has to be $\sigma : S \to A$, a total map of event structures with polarity, such that
(i) whenever $\sigma x \subseteq^- y$ in $\mathcal{C}(A)$ there is a unique $x' \in \mathcal{C}(S)$ so that
$x \subseteq x'$ & $\sigma x' = y$, *i.e.*

$$\begin{tikzcd} x \arrow[r, dotted, phantom, "\subseteq"] \arrow[d, "\sigma"'] & x' \arrow[d, dotted, "\sigma"] \\ \sigma x \arrow[r, phantom, "\subseteq^-"] & y \end{tikzcd}$$

and

(ii) whenever $y \subseteq^+ \sigma x$ in $\mathcal{C}(A)$ there is a (necessarily unique) $x' \in \mathcal{C}(S)$ so that
$x' \subseteq x$ & $\sigma x' = y$, *i.e.*

$$\begin{tikzcd} x' \arrow[r, dotted, phantom, "\subseteq"] \arrow[d, dotted, "\sigma"'] & x \arrow[d, "\sigma"] \\ y \arrow[r, phantom, "\subseteq^+"] & \sigma x \end{tikzcd}$$

The only immediate causal dependencies a strategy can introduce: $\ominus \rightarrowtriangle \oplus$

A bicategory of games

Objects are event structures with polarity—the games, A, B, ... ;
Arrows $\sigma : A \nrightarrow B$ are strategies $\sigma : S \to A^\perp \| B$;

2-Cells $\begin{tikzcd} A \arrow[r, bend left, "\sigma"{name=U}] \arrow[r, bend right, "\sigma'"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "f"] \end{tikzcd}$ are maps $f : S \to S'$ such that $\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "\sigma"'] \arrow[dr, phantom, shift left=3, "="] & S' \arrow[d, "\sigma'"] \\ & A^\perp \| B \end{tikzcd}$.

The vertical composition of 2-cells is the usual composition of maps. Horizontal composition is given by $\odot$ (which extends to a functor via universality).

Full sub-bicategory when games are purely +ve: 'stable spans' used in nondeterministic dataflow—feedback is given by trace; when strategies are deterministic, Berry's **dI-domains and stable functions***, and its subcategories of Girard's* **coherence spaces** *and* **qualitative domains***.* **Scott domains?**

Strategies as profunctors

A strategy in a game A is a (special) presheaf over the configurations $\mathcal{C}(A)$.

A strategy from A to B is a (special) profunctor from $\mathcal{C}(A)$ to $\mathcal{C}(B)$.

Recall,
a **presheaf** over a (partial order) category $\mathbb{A}$ is a functor from $\mathbb{A}^{\mathrm{op}}$ to $\mathbf{Set}$.

It corresponds to a **discrete fibration** $F : \mathbb{S} \to \mathbb{A}$, $\exists! x'$.

$$\begin{tikzcd} x' \arrow[r, phantom, "\sqsubseteq_{\mathbb{S}}", dotted] \arrow[d, mapsto, dotted, "F"'] & x \arrow[d, mapsto, "F"] \\ y \arrow[r, phantom, "\sqsubseteq_{\mathbb{A}}"] & Fx \,. \end{tikzcd}$$

A **profunctor** from a category $\mathbb{A}$ to $\mathbb{B}$ is a presheaf over $\mathbb{A}^{\mathrm{op}} \times \mathbb{B}$.

When replace $\mathbf{Set}$ *by* $0 < 1$,
presheaves become down-closed sets and profunctors become relations between partial orders, cf. approximable mappings.

Recall the definition of strategy

A **strategy** in a game A is $\sigma : S \to A$, a total map of event structures with polarity, such that

(i) whenever $\sigma x \subseteq^{-} y$ in $\mathcal{C}(A)$ there is a unique $x' \in \mathcal{C}(S)$ so that $x \subseteq x' \ \&\ \sigma x' = y$, *i.e.*

$$\begin{tikzcd} x \arrow[r, dotted, phantom, "\subseteq"] \arrow[d, mapsto, "\sigma"'] & x' \arrow[d, dotted, mapsto, "\sigma"] \\ \sigma x \arrow[r, phantom, "\subseteq^{-}"] & y\,, \end{tikzcd}$$

and

(ii) whenever $y \subseteq^{+} \sigma x$ in $\mathcal{C}(A)$ there is a (necessarily unique) $x' \in \mathcal{C}(S)$ so that $x' \subseteq x \ \&\ \sigma x' = y$, *i.e.*

$$\begin{tikzcd} x' \arrow[r, dotted, phantom, "\subseteq"] \arrow[d, dotted, mapsto, "\sigma"'] & x \arrow[d, mapsto, "\sigma"] \\ y \arrow[r, phantom, "\subseteq^{+}"] & \sigma x\,. \end{tikzcd}$$

An alternative characterization of strategies

Defining a partial order — *the Scott order* — on configurations of A

$$y \sqsubseteq_A x \ \text{ iff } \ y \supseteq^- \cdot \subseteq^+ \cdot \supseteq^- \cdots \supseteq^- \cdot \subseteq^+ x$$

we obtain a factorization system $((\mathcal{C}(A), \sqsubseteq_A), \supseteq^-, \subseteq^+)$, $i.e.$ $\exists! z.$

$$\begin{tikzcd} & x \\ y \arrow[ur, dotted, "\sqsubseteq"] & z \arrow[u, phantom, "\cup\!\!\mid^{+}"] \arrow[l, phantom, "\supseteq^-"] \end{tikzcd}.$$

Proposition $z \in \mathcal{C}(\mathbb{C}_A)$ iff $z_2 \sqsubseteq_A z_1$.

Theorem Strategies $\sigma : S \to A$ correspond to discrete fibrations

$$\sigma`` : (\mathcal{C}(S), \sqsubseteq_S) \to (\mathcal{C}(A), \sqsubseteq_A)\,, \quad i.e. \quad \exists! x'. \quad \begin{tikzcd} x' \arrow[r, dotted, dash, "\sqsubseteq_S"] \arrow[d, dotted, "\sigma``"'] & x \arrow[d, "\sigma``"] \\ y \arrow[r, phantom, "\sqsubseteq_A"] & \sigma``(x) \end{tikzcd},$$

which preserve $\supseteq^-$, $\subseteq^+$ and $\emptyset$.

From strategies to profunctors

A strategy σ from A to B determines a discrete fibration so a presheaf over

$$\begin{aligned}(\mathcal{C}(A^\perp \| B), \sqsubseteq_{A^\perp \| B}) &\cong (\mathcal{C}(A^\perp), \sqsubseteq_{A^\perp}) \times (\mathcal{C}(B), \sqsubseteq_B) \\ &\cong (\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)\end{aligned}$$

$i.e.$ a profunctor $\sigma`` : (\mathcal{C}(A), \sqsubseteq_A) \nrightarrow (\mathcal{C}(B), \sqsubseteq_B)$.

$\rightsquigarrow$ a lax pseudo functor $(\_)`` : \mathbf{Games} \to \mathbf{Prof}$; have $(\tau \odot \sigma)`` \Rightarrow \tau`` \circ \sigma``$.
The profunctor composition introduces extra 'unreachable' elements.

Laxness prompts: What's missing in categories and profunctors?
$\rightsquigarrow$ *games as 'rooted' factorisation systems, strategies as 'rooted' profunctors.*

Games as factorisation systems

A **rooted factorisation system** $(\mathbb{C}, L, R, 0)$ comprises a small category $\mathbb{C}$ on which there is a factorisation system $(\mathbb{C}, L, R)$,

so all maps $c \to c'$ factor uniquely up to iso as

$$\begin{tikzcd} & c' \\ c \arrow[ur] \arrow[r, "L"'] & c'' \arrow[u, "R"'] \end{tikzcd},$$

with an object 0 s.t. for all objects c in $\mathbb{C}$, there is a path

$$0 \leftarrow_L \cdot \rightarrow_R \cdots \leftarrow_L \cdot \rightarrow_R c\,,\ \text{with no nontrivial paths to } 0,$$

$$\begin{tikzcd} & \cdot \arrow[dr, dotted, "L"] & \\ \cdot \arrow[ur, "L"] \arrow[dr, "L"'] & & \cdot \\ & \cdot \arrow[ur, dotted, "L"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \cdot \arrow[dr, "R"] & \\ \cdot \arrow[ur, dotted, "R"] \arrow[dr, dotted, "R"'] & & \cdot \\ & \cdot \arrow[ur, "R"'] & \end{tikzcd}$$

E.g. $(\,(\mathcal{C}(A), \sqsubseteq_A)\ ,\ \supseteq^-, \subseteq^+, \emptyset)$.

Strategies

A strategy on a rooted factorization system $(\mathbb{A}, L_A, R_A, 0_A)$ is a discrete fibration

$$F : (\mathbb{S}, L_S, R_S, 0_S) \to (\mathbb{A}, L_A, R_A, 0_A)\,,$$

from another rooted factorization system $(\mathbb{S}, L_S, R_S, 0_S)$, which preserves L, R maps and 0.

Example: The map $\sigma`` : ((\mathcal{C}(S), \sqsubseteq_S), \supseteq^-, \subseteq^+, \emptyset) \to ((\mathcal{C}(A), \sqsubseteq_A), \supseteq^-, \subseteq^+, \emptyset)$ induced by a strategy $\sigma : S \to A$.

Operations $\quad (\mathbb{C}, L, R, 0)^\perp =_{\rm def} (\mathbb{C}^{\rm op}, R^{\rm op}, L^{\rm op}, 0)$

$(\mathbb{B}, L_B, R_B, 0_B) \| (\mathbb{C}, L_C, R_C, 0_C) =_{\rm def} (\mathbb{B} \times \mathbb{C}, L_B \times L_C, R_B \times R_C, (0_B, 0_C))$

Composition: *reachable part of* profunctor composition.

Games and strategies embed fully and faithfully in rooted factorization systems.

Bidomains

Berry's **bidomains**: $(D, \leq, \sqsubseteq)$ with functions continuous w.r.t. $\sqsubseteq$ and stable w.r.t. $\leq$. *Represented by bistructures* $(E, \leq_L, \leq_R, \#)$ *[1980].*

Defining $\sqsubseteq^R = \leq$ and

$$x \sqsubseteq^L y \iff x \sqsubseteq y \ \& \ (\forall z \in D.\ (x \sqsubseteq z \ \& \ z \sqsubseteq_R y) \Rightarrow y = z)\,,$$

a bidomain corresponds to a rooted factorisation system $(D, \sqsubseteq_L, \sqsubseteq_R, \bot)$ provided

$$x \downarrow^L y \ \Rightarrow \ x \uparrow^L y\,.$$

Preserved by function space?!

Such **rooted** bidomains embed faithfully in rooted factorisation systems.
Fully in deterministic strategies of rooted factorisation systems?

Some unfinished business

• Bidomains?

• How's the "factorisation story" affected by non-linearity?
Non-linearity via event structures with symmetry.
The Scott order becomes a Scott category.
Strategies as certain fibrations - a characterisation?

• A curiosity?
The Scott order is a bottomless cpo. Algebraic? Not countable basis.

The influences from domain theory to concurrent games

... are numerous, from broad methodology to specific definitions,

E.g. The definition of *probabilistic strategies* depends on probabilistic event structures; essentially event structures with a continuous valuation on the Scott open sets. A characterisation via a "drop condition," a condition on the probabilities assigned to finite configurations.

The "drop" condition on operators is key to the extension to *quantum strategies*.

LICS'18: *Full abstraction for probabilistic PCF via probabilistic strategies with symmetry* – with Simon Castellan, Pierre Clairambault and Hugo Paquet.

Domain theory is here to stay!
Why use a complicated model when a simple model will do?