$\mathbb{T}^\omega$-representations of compact sets through dyadic subbases

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Represented Space

- Represented space $\mathbf{X} = (X, \delta_X)$:
  - δ_X : partial surjective function from A to X. ($A = \underline{\{0,1\}^\omega}, \mathbb{N}^\omega, \ldots$).
  - Lift the computability notion in A to X.

$$\begin{tikzcd} \mathbf{X} \arrow[r, rightrightarrows, "f"] & \mathbf{Y} \\ A \arrow[u, "\delta_X"] \arrow[r, "F"] & A \arrow[u, "\delta_Y"'] \end{tikzcd}$$

f is computable if $\exists$ computable F.

- Sierpinski space $\mathbb{S} = \{\top, \bot\}$. $(\mathbb{S}, \delta_{\mathbb{S}})$ is a represented space.
  - $\delta_{\mathbb{S}}(0^\omega) = \bot$
  - $\delta_{\mathbb{S}}(0^n 1 p) = \top$
- Plotkin's $\mathbb{T} = \{0, 1, \bot\}$. $(\mathbb{T}, \delta_{\mathbb{T}})$ is a represented space.
  - $\delta_{\mathbb{T}}(0^\omega) = \bot$
  - $\delta_{\mathbb{T}}(0^{2n} 1 p) = 0$
  - $\delta_{\mathbb{T}}(0^{2n+1} 1 p) = 1$
- $(\mathbb{T}^\omega, \delta_{\mathbb{T}^\omega})$ is a represented space.
- We have a derived computability notion on $\mathbb{T}^\omega$.

$$\begin{tikzcd} \mathbb{T}^\omega \arrow[r, rightrightarrows, "f"] & \mathbb{T}^\omega \\ \{0,1\}^\omega \arrow[u, "\delta_{\mathbb{T}^\omega}"] \arrow[r, "F"] & \{0,1\}^\omega \arrow[u, "\delta_{\mathbb{T}^\omega}"'] \end{tikzcd}$$

$\mathbb{T}^\omega$-Represented Space

- $\mathbb{T}^\omega$-Represented space (X, ψ):
  - ψ : partial surjective function from $\mathbb{T}^\omega$ to X.
  - Lift the computability notion in $\mathbb{T}^\omega$ to X.
  - Note that we consider multi-valued realizers on $\mathbb{T}^\omega$.

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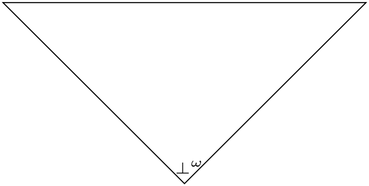
  - We can expect a natural representation of a space with order.



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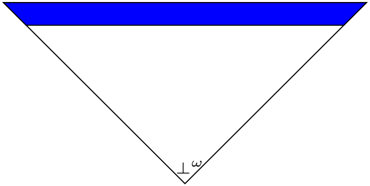

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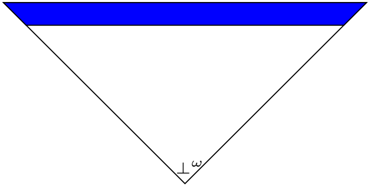

$\perp$	0	1	$\perp$	1	$\cdots$

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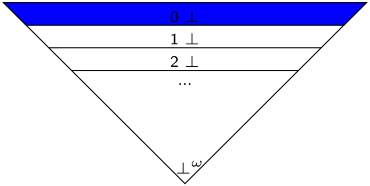
- Contains $\{0,1\}^\omega$ as top elements.
  - Sub-structure of the space can be represented with $\{0,1\}^\omega$.
- Direct manipulation of a sequence of partial information.
- Can count the number of bottoms.
  - Some information of the space can be reflected into the number of bottoms.
  - Every *n*-dimensional second countable metrizable space can be injectively represented in $\mathbb{T}^\omega$ so that each name contains up to *n* copies of $\perp$ [T 2002].



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$\mathbb{T}^\omega$-sequences as unspecified points.

Suppose that $\delta :\subseteq \{0,1\}^\omega \to \mathbf{X}$ is a representation.

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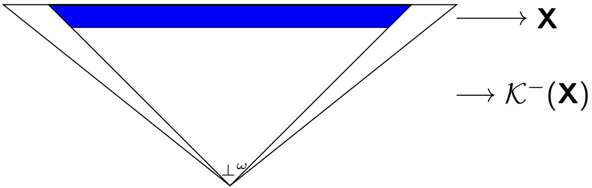
  - Yes, for $\mathbf{X}$ a computably compact computable metric space.

Matching representation.

Let $\delta :\subseteq \{0,1\}^\omega \to \mathbf{X}$ and $\psi :\subseteq \mathbb{T}^\omega \to \mathcal{K}^-(\mathbf{X})$ be ($\mathbb{T}^\omega$-)representations.



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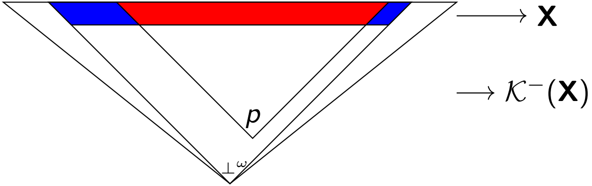

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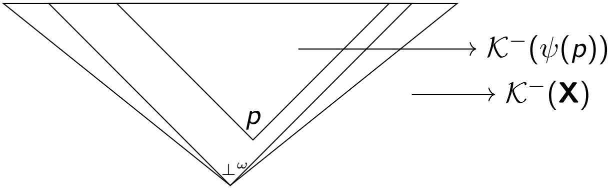

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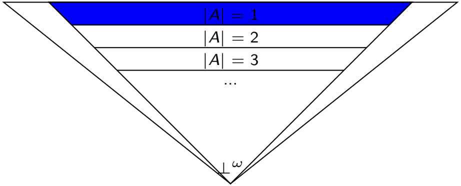

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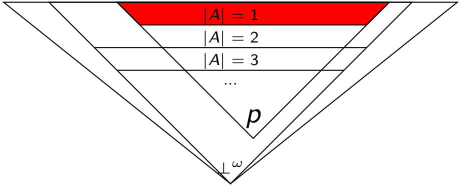
- ψ is faithful if it is hereditary and minimal.
    - If ψ is faithful, then (δ, ψ) match for $\delta = \psi|_{\{0,1\}^\omega}$.



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Theorem (Main Theorem)

If $\mathbf{X}$ is a CCCMS (computably compact computable metric space), then $\mathcal{K}^-(\mathbf{X})$ admits a faithful $\mathbb{T}^\omega$-representation.

Note1: Applications in finite closed choice and Weihrauch reducibility.

Note2: Compact = closed in a compact metric space. $\mathcal{K}(\mathbf{X}) \Leftrightarrow \mathcal{A}(\mathbf{X})$.

Outline of the proof

Computably compact computable metric space

$\Downarrow$

Existence of proper computable dyadic subbases

$\Downarrow$

Domain representation

$\Downarrow$

Compact sets are tree-like subsets

$\Downarrow$ (Expanding tree-like sets as trees)

Cantor space (Compact sets are binary trees)

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Faithful representation

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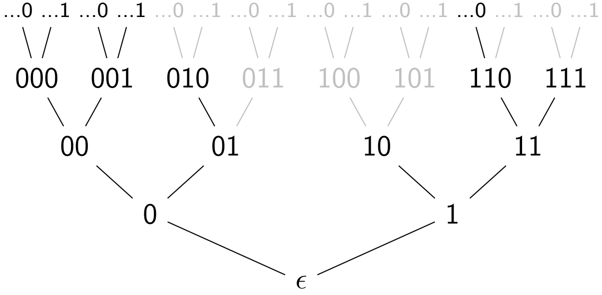
Closed subsets of Cantor space as binary trees.

- A binary tree ($\mathcal{T}$) is a prefix-closed subset of $\{0,1\}^*$.
- Infinite paths of $T \in \mathcal{T}$ form a closed subset $[T]$ of $\{0,1\}^\omega$.



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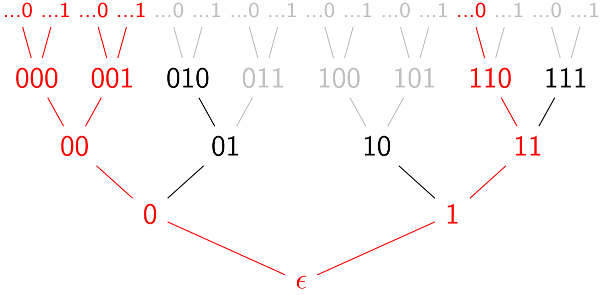
- Infinite paths of $T \in \mathcal{T}$ form a closed subset $[T]$ of $\{0,1\}^\omega$.
- A pruned tree ($\mathcal{PT}$) is a tree T such that $w \in T$ implies $\exists v \in T\ w \sqsubset v$.
- $[\cdot]$ is a one to one correspondence between $\mathcal{PT}$ and $\mathcal{A}(\{0,1\}^\omega)$ (Closed subsets of $\{0,1\}^\omega$).
- Consider $\mathbb{T}^\omega$-representation of pruned trees.

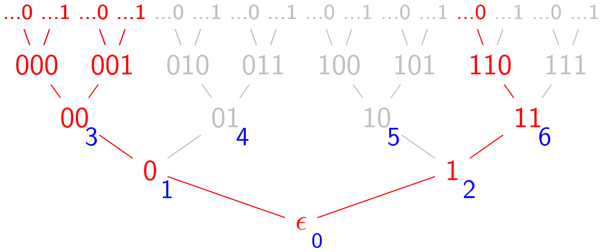


$\mathbb{T}^\omega$-representation $\delta_{\mathcal{PT}} : \mathbb{T}^\omega \to \mathcal{PT}$ of Pruned Trees

Let $n(p)$ for $p \in \{0,1\}^*$ be the enumeration of $\{0,1\}^*$ as blue.
For a pruned tree T,

$$\delta_{\mathcal{PT}}^{-1}(T)(n(p)) = \begin{cases} \perp & (\text{both } 0p, 1p \in T) \\ 0 & (0p \in T) \\ 1 & (1p \in T) \\ 0 \text{ or } 1 & (p \notin T (\text{causes redundancy})) \end{cases}$$

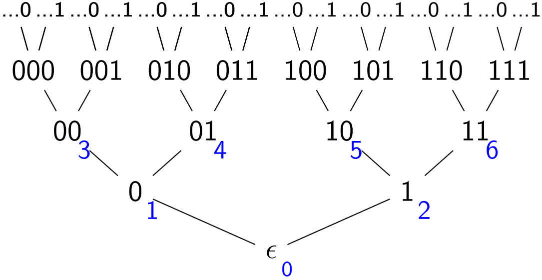
Neither do not happen for a pruned tree.



$\perp$	0	1	$\perp$	$*$	$*$	0	$\perp$	$\perp$	$*$	$*$	$*$	$*$	0	$*$	$\cdots$
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	

Computing $\delta_{\mathcal{PT}}$-name incrementally

Let A be the tree $\{0,1\}^* \setminus (\uparrow p_1 \cup \uparrow p_1 \cup \ldots)$.
Given a sequence $p_1, p_2, \ldots \in \{0,1\}^*$ (as a $\{0,1\}^*$-name of a closed set) compute a $\delta_{\mathcal{PT}}$-name of $Prune(A)$.



$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	...
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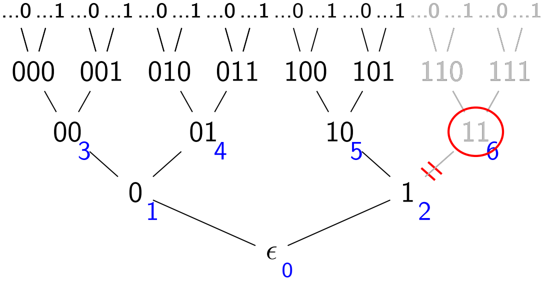

1. Detected $\uparrow 11 \not\subseteq A$.

$\perp$	$\perp$	0	$\perp$	$\perp$	$\perp$	0	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	0	0	$\cdots$
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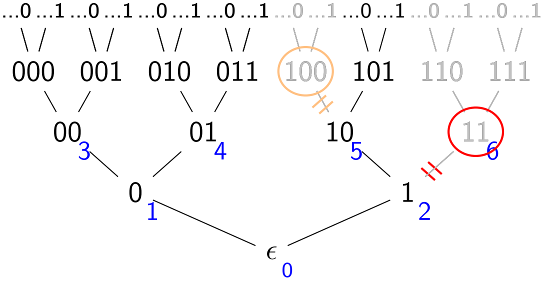

1. Detected $\uparrow 11 \notin A$.
2. Detected $\uparrow 100 \notin A$.

$\perp$	$\perp$	0	$\perp$	$\perp$	1	0	$\perp$	$\perp$	$\perp$	$\perp$	0	$\perp$	0	0	...
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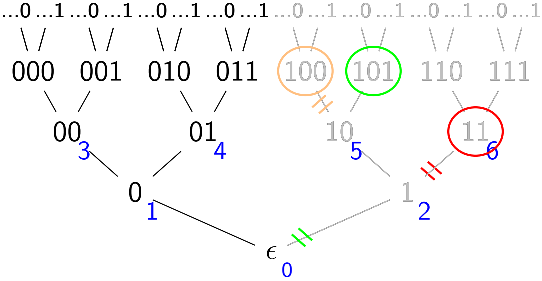

1. Detected $\uparrow 11 \notin A$.
2. Detected $\uparrow 100 \notin A$.
3. Detected $\uparrow 101 \notin A$.

0	$\perp$	0	$\perp$	$\perp$	1	0	$\perp$	$\perp$	$\perp$	$\perp$	0	0	0	0	⋯
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The number of bottoms

- Let $\mathcal{PT}$ be the $\mathbb{T}^\omega$-represented space $(\mathcal{PT}, \delta_{\mathcal{PT}})$.
- $\delta_{\mathcal{T}} : (\{0,1\}^*)^\omega \to \mathbb{T}$, $p_1, p_2, \ldots \mapsto \{0,1\}^* \setminus (\uparrow p_1 \cup \uparrow p_1 \cup \ldots)$ is a $(\{0,1\}^*)^\omega$-representation. $\mathcal{T} = (\mathcal{T}, \delta_{\mathcal{T}})$.
- The above computation shows that $Prune : \mathcal{T} \to \mathcal{PT}$ is computable.
- Similarly, one can show that its multi-valued inverse is computable.
- $[\cdot] : \mathcal{T} \to \mathcal{A}(\{0,1\}^\omega)$ is computable with computable multi-valued inverse.
- $Prune \circ [\cdot]^{-1}$ is a computable isomorphism between $\mathcal{K}(\{0,1\}^\omega) = \mathcal{A}(\{0,1\}^\omega)$ and $\mathcal{PT}$, and obtain a $\mathbb{T}^\omega$-representation of $\mathcal{K}(\{0,1\}^\omega)$.
- It is a hereditary representation.
- If A is a finite set of cardinarity n, only $n-1$ number of $\perp$ is left (minimal).

Theorem

$\mathcal{K}^-(\{0,1\}^\omega)$ has a faithful $\mathbb{T}^\omega$-representation.

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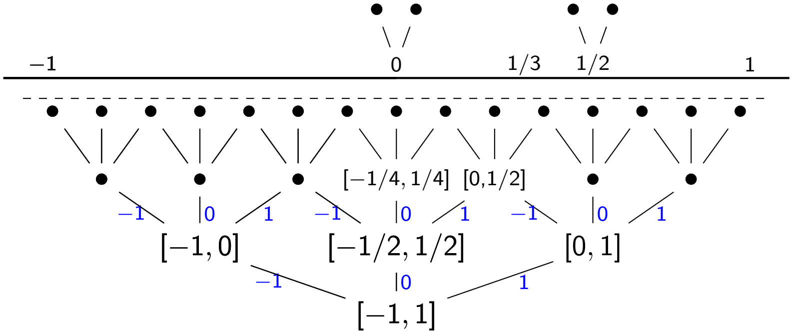
Faithful representation

Unit Interval as Retract of the limit-set of a DAG

- Signed digit representation of the unit interval.
- Three-branching, two-merging DAG (Directed Acyclic Graph).
- Infinite paths (obtained as the ideal completion) are representing the unit interval.
- Note: three essentially different paths represent 0.



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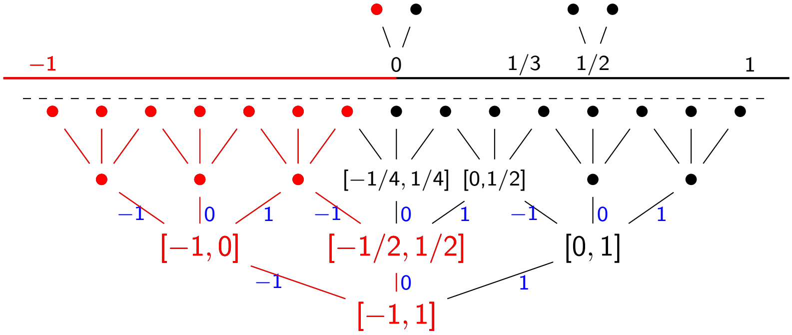
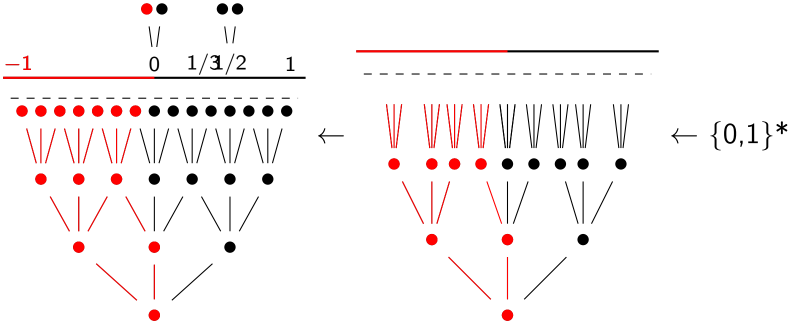
- Tree-like subsets ($\forall p \neq \epsilon \in A\,(\exists q \in A\ q \preceq^1 p)$) of this DAG are representing closed subsets.



Expansion of a DAG to a tree



Tree-like set $\longleftarrow$ Tree $\leftarrow$ Binary Tree $\leftarrow$ Pruned Binary Tree

$\uparrow \delta_{\mathcal{PT}}$

$\mathbb{T}^\omega$

Question: Which space can be represented as the limit-set of a Scott domain that is obtained as the ideal completion of this kind of DAG?

Outline of the proof

Computably compact computable metric space

$\Downarrow$

Existence of proper computable dyadic subbases

$\Downarrow$

Domain representation

$\Downarrow$

Compact sets are tree-like subsets

$\Downarrow$ (Expanding tree-like sets as trees)

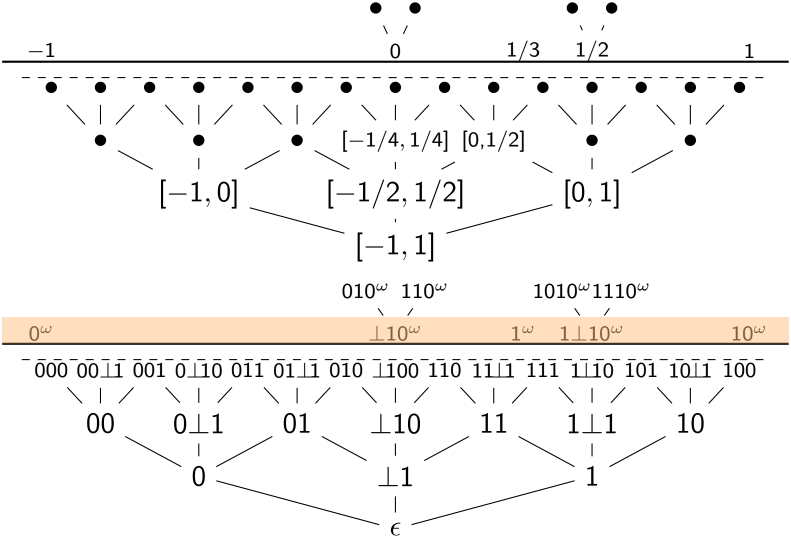
Cantor space (Compact sets are binary trees)

$\Downarrow$

Faithful representation

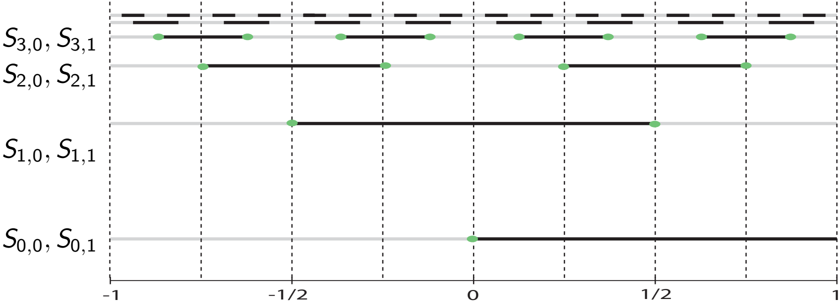
Gray code [Gianantonio:1999],[T:2002]

$G : [-1, 1] \to \mathbb{T}^\omega$: Injective $\mathbb{T}^\omega$-representation of the unit interval.



How is it constructed?

$S_{n,0} = \{x \mid G(x)(n) = 0\}$, $S_{n,1} = \{x \mid G(x)(n) = 1\}$,



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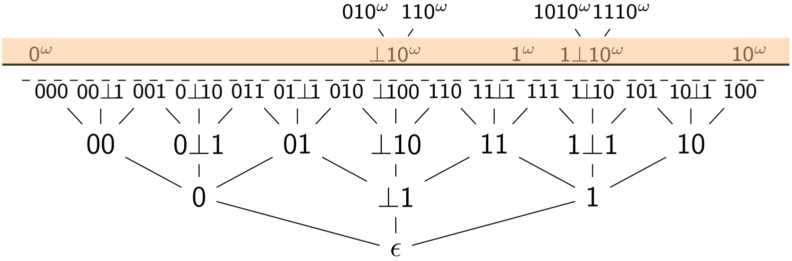
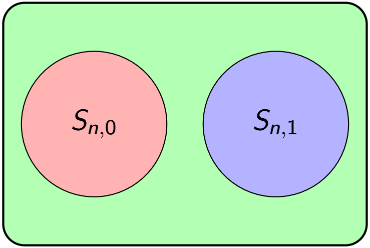


Dyadic subbase — Generalization of Gray-code.

Definition ([T 2004])

A *dyadic subbase* over a set X is a map $S : \mathbb{N} \times \{0,1\} \to \mathcal{P}(X)$ such that $S_{n,0} \cap S_{n,1} = \emptyset$ for every $n \in \mathbb{N}$ and
$\{(n,i) \mid x \in S_{n,i}\} = \{(n,i) \mid y \in S_{n,i}\} \Rightarrow x = y$ for $x, y \in X$.

- $S_{n,\perp} = X \setminus (S_{n,0} \cup S_{n,1})$.
- $\varphi_S(x)(n) = \begin{cases} 0 & (x \in S_{n,0}), \\ 1 & (x \in S_{n,1}), \\ \perp & (x \in S_{n,\perp}). \end{cases}$
- $\varphi_S : X \to \mathbb{T}^\omega$: embedding into $\mathbb{T}^\omega$.
- $\mathbf{X}_S = (X, \varphi_S^{-1})$ is an admissible $\mathbb{T}^\omega$-represented space.
- We say that S is a computable dyadic subbase of a represented space $\mathbf{X}$ if $\mathbf{X}_S$ is computably isomorphic to $\mathbf{X}$.



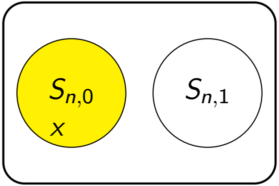
n-th coordinate

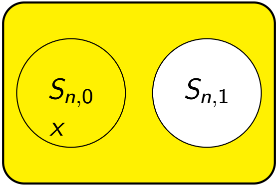
Two kinds of information.

- Each finite sequence $p \in \mathbb{T}^*$ specifies

$$S(p) = \bigcap_{n \in \mathrm{dom}(p)} S_{n,p(n)}$$

$$\bar{S}(p) = \bigcap_{n \in \mathrm{dom}(p)} (X \setminus S_{n,1-p(n)}) = \bigcap_{n \in \mathrm{dom}(p)} (S_{n,p(n)} \cup S_{n,\perp})$$





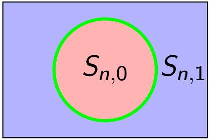
Definition

A dyadic subbase S is proper if $\bar{S}(p) = \mathrm{cl}\, S(p)$ for every $p \in \mathbb{T}^*$.

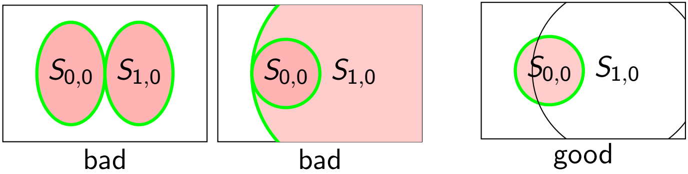
Proper dyadic subbase

Recall that S is proper means $\bar{S}(p) = \mathrm{cl}\, S(p)$ for every $p \in \mathbb{T}^*$.

- Gray-code is a proper dyadic subbase.
- $S_{n,0}$ and $S_{n,1}$ are exteriors of each other. (The case $p = \perp^n 1$.) $S_{n,\perp}$ is the common boundary.



- $S_{0,\perp}$ and $S_{1,\perp}$ do not touch. (The case $p = 00$.)



Computability notions of $\bar{S}(p)$ and $\mathrm{cl}\, S(p)$

Two computability notions $\mathcal{A}(\mathbf{X})$ (closed) and $\mathcal{V}(\mathbf{X})$ (overt) for closed sets.

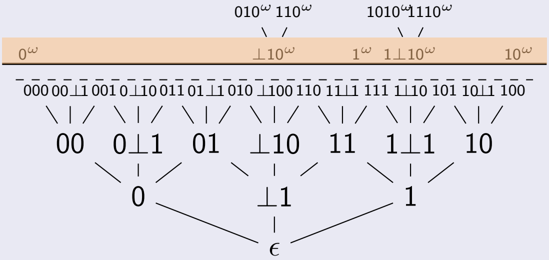
- $\bar{S}(p) \in \mathcal{A}(\mathbf{X})$ because $\bar{S}(p) = X \setminus (\bigcup_{n \in \mathrm{dom}(p)} S_{n,1-p(n)})$.
  - $A \in \mathcal{A}(\mathbf{X}) \iff A^C \in \mathcal{O}(\mathbf{X})$.
  - Representation by negative information.
- $\mathrm{cl}\, S(p) \in \mathcal{V}(\mathbf{X})$ because $\mathrm{cl}\, S(p) = \mathrm{cl}\,(\bigcap_{n \in \mathrm{dom}(p)} S_{n,p(n)})$.
  - $A \in \mathcal{V}(\mathbf{X})$ is represented by enumeration of $\{U \mid U \cap A \neq \emptyset\}$.
  - Representation by positive information.
- If S is a proper dyadic subbase, then $\bar{S}(p) = \mathrm{cl}\, S(p) \in \mathcal{V}(\mathbf{X}) \wedge \mathcal{K}(\mathbf{X})$.
- $\{\bar{S}(p) \mid p \in \mathbb{T}^*\}$ is a closed base which consists of overt and closed sets.

Let S be a dyadic subbase of a space $\mathbf{X}$. Define

$$\widehat{K}_S = \{p_{\langle \leq n \rangle} \mid \exists x \in X \ \varphi_S(x) \preceq p, n \in \mathbb{N}\}$$
$$\widehat{D}_S = \text{the ideal completion of } \widehat{K}_S \ (\subseteq \mathbb{T}^\omega), \quad \widehat{L}_S = \widehat{D}_S \setminus \widehat{K}_S.$$

Theorem (T,Tsukamoto,2015)

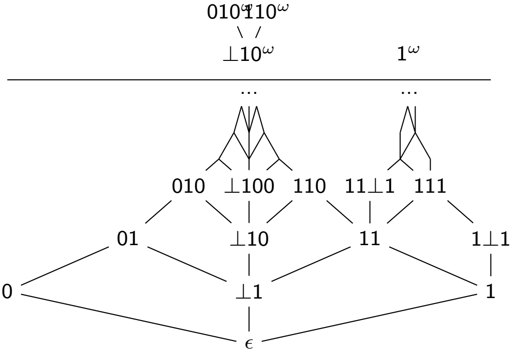
Suppose that S is a proper computable dyadic subbase of a CCCMS X. We have something like



$\widehat{K}_S$: *Finite branching DAG.*
$\widehat{D}_S$: *Scott Domain*
$\min(\widehat{L}_S)$: *Homeomorphic to X*
$\widehat{L}_S \to X$: *Computable retract*

By expanding it into tree, we have a $\mathbb{T}^\omega$-representation of the set $\mathcal{K}^-(\mathbf{X})$. However, it is not computable because $\widehat{K}_S$ is not computable.

To obtain a faithful $\mathbb{T}^\omega$-representation of $\mathcal{K}^-(\mathbf{X})$ from a proper computable dyadic subbase, we consider a bigger domain and construct an algorithm to restrict the limits to $\widehat{L}_S$ using the fact that $p \in \widehat{K}_S$ is semi-decidable. (Omit the details.)



This is a tree-like set representing the set $\{0, 1/3\}$. Therefore, we need to assign a $\mathbb{T}^\omega$ sequence with one $\perp$.

Theorem

If S is a proper computable dyadic subbase of a CCCMS X, then X has a faithful $\mathbb{T}^\omega$-representation.

Outline of the proof

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$\Downarrow$ (Expanding tree-like sets as trees)

Cantor space (Compact sets are binary trees)

$\Downarrow$

Faithful representation

Construction of a proper computable dyadic subbase

Theorem: Every separable metric space has a proper dyadic subbase.

- Proved in [Ohta, Yamada, T 2011] for a special case and in [Ohta, Yamada, T 2013] for the full case.
- Tsukamoto gave an elegant proof in [Tsukamoto 2017]. (In that paper, he also proved that every locally compact separable metric space has a strongly proper dyadic subbase.) We effectized his proof to obtain

Theorem

Every CCCMS has a computable proper dyadic subbase.

Theorem (main theorem)

If $\mathbf{X}$ *is a CCCMS, then* $\mathcal{K}^{-}(\mathbf{X})$ *has a faithful* $\mathbb{T}^{\omega}$*-representation.*

Thank you very much.