$\mathbb{T}^\omega$-representations of compact sets through dyadic subbases

Arno Pauly and Hideki Tsuiki

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Represented Space

- Represented space $\mathbf{X} = (\mathbf{X}, \delta_X)$:
  - $\delta_X$ : partial surjective function from $A$ to $\mathbf{X}$. ($A = \{0, 1\}^\omega, \mathbb{N}^\omega, \ldots$).
  - Lift the computability notion in $A$ to $\mathbf{X}$.

\[
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{f} & \mathbf{Y} \\
\delta_X \uparrow & & \delta_Y \\
A & \xrightarrow{F} & A
\end{array}
\]

- $f$ is computable if $\exists$ computable $F$.

- Sierpinski space $\mathbb{S} = \{\top, \bot\}$. ($\mathbb{S}, \delta_{\mathbb{S}}$) is a represented space.

  - $\delta_{\mathbb{S}}(0^\omega) = \bot$
  - $\delta_{\mathbb{S}}(0^n1p) = \top$

- Plotkin’s $\mathbb{T} = \{0, 1, \bot\}$. ($\mathbb{T}, \delta_{\mathbb{T}}$) is a represented space.

  - $\delta_{\mathbb{T}}(0^\omega) = \bot$
  - $\delta_{\mathbb{T}}(0^2n1p) = 0$
  - $\delta_{\mathbb{T}}(0^2n+11p) = 1$

- ($\mathbb{T}^\omega, \delta_{\mathbb{T}^\omega}$) is a represented space.

- We have a derived computability notion on $\mathbb{T}^\omega$. $\{0, 1\}^\omega \xrightarrow{F} \{0, 1\}^\omega$
$\mathbb{T}^\omega$-Represented Space

- $\mathbb{T}^\omega$-Represented space $(X, \psi)$:
  - $\psi$: partial surjective function from $\mathbb{T}^\omega$ to $X$.
  - Lift the computability notion in $\mathbb{T}^\omega$ to $X$.
  - Note that we consider multi-valued realizers on $\mathbb{T}^\omega$.

```
\begin{array}{c}
\text{X} & \overset{f}{\longrightarrow} & \text{Y} \\
\psi_X & \uparrow & \psi_Y \\
\mathbb{T}^\omega & \overset{F}{\longrightarrow} & \mathbb{T}^\omega
\end{array}
```
\( \mathbb{T}^\omega \)-Represented Space

- \( \mathbb{T}^\omega \)-Represented space \((X, \psi)\):
  - \( \psi : \) partial surjective function from \( \mathbb{T}^\omega \) to \( X \).
  - Lift the computability notion in \( \mathbb{T}^\omega \) to \( X \).
  - Note that we consider multi-valued realizers on \( \mathbb{T}^\omega \).

\[
\begin{array}{c}
\xymatrix{
X \ar[r]^f & Y \\
\mathbb{T}^\omega \ar[u]^{\psi_X} \ar[r]^{F} & \mathbb{T}^\omega \ar[u]^{\psi_Y} \\
\{0, 1\}^\omega \ar[r]^G & \{0, 1\}^\omega
}
\end{array}
\]
$\mathbb{T}^\omega$-Represented Space

- $\mathbb{T}^\omega$-Represented space $(X, \psi)$:
  - $\psi$ : partial surjective function from $\mathbb{T}^\omega$ to $X$.
  - Lift the computability notion in $\mathbb{T}^\omega$ to $X$.
  - Note that we consider multi-valued realizers on $\mathbb{T}^\omega$.

- Why $\mathbb{T}^\omega$-represented space?
Why $\mathbb{T}^\omega$-represented space?

- $(\mathbb{T}^\omega, \preceq)$ has an order structure. It is a Scott domain!
  - We can expect a natural representation of a space with order.
Why $T^\omega$-represented space?

- $(T^\omega, \preceq)$ has an order structure. It is a Scott domain!
  - We can expect a natural representation of a space with order.

- Contains $\{0, 1\}^\omega$ as top elements.
  - Sub-structure of the space can be represented with $\{0, 1\}^\omega$. 

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Why $T^\omega$-represented space?

- $(T^\omega, \preceq)$ has an order structure. It is a Scott domain!
  - We can expect a natural representation of a space with order.
- Contains $\{0, 1\}^\omega$ as top elements.
  - Sub-structure of the space can be represented with $\{0, 1\}^\omega$.
- Direct manipulation of a sequence of partial information.

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
\perp & 0 & 1 & \perp & 1 & \cdots \\
\end{array}
\]
Why $\mathbb{T}^\omega$-represented space?

- $(\mathbb{T}^\omega, \leq)$ has an order structure. It is a Scott domain!
  - We can expect a natural representation of a space with order.
- Contains $\{0, 1\}^\omega$ as top elements.
  - Sub-structure of the space can be represented with $\{0, 1\}^\omega$.
- Direct manipulation of a sequence of partial information.
- Can count the number of bottoms.
  - Some information of the space can be reflected into the number of bottoms.
  - Every $n$-dimensional second countable metrizable space can be injectively represented in $\mathbb{T}^\omega$ so that each name contains up to $n$ copies of $\bot$ [T 2002].

![Diagram](image.png)
\( \mathbb{T}^\omega \)-sequences as unspecified points.

Suppose that \( \delta : \subseteq \{0, 1\}^\omega \rightarrow X \) is a representation.

- Each infinite sequence (in \( \text{dom}(\delta) \)) specifies a point.
  - \( 0101101... \mapsto x \in X \)

Can we use it as a \( \mathbb{T}^\omega \)-representation of \( \mathcal{P}(X) \)?

- No. (Cardinality).

Can we use it as a \( \mathbb{T}^\omega \)-representation of the set \( \mathcal{K}(X) \) of non-empty compact subsets of \( X \)?

- Yes, for \( X \) a computably compact computable metric space.
Suppose that $\delta : \subseteq \{0, 1\}^\omega \to X$ is a representation.

- Each infinite sequence (in $\text{dom}(\delta)$) specifies a point.
  
  $\triangleright$ $0101101... \mapsto x \in X$

- Each $T$-sequence specifies an unspecified point.

  $\triangleright$ $01\_\_0\_\_1\_\_... \mapsto \begin{cases} 
  \delta(010001...), \\
  \delta(010011...), \\
  \delta(011001...), \\
  \delta(011011...) 
\end{cases}$ (If all of them are in $\text{dom}(\delta)$)
\textbf{\(T^\omega\)-sequences as unspecified points.}

Suppose that \(\delta : \subseteq \{0, 1\}^\omega \to X\) is a representation.

- Each infinite sequence (in \(\text{dom}(\delta)\)) specifies a point.
  - \(0101101... \mapsto x \in X\)

- Each \(T\)-sequence specifies an unspecified point.
  - \(01\perp 0\perp 1... \mapsto \begin{cases} \delta(010001...), \\ \delta(010011...), \\ \delta(011001...), \\ \delta(011011...) \end{cases}\) (If all of them are in \(\text{dom}(\delta)\))

- Can we use it as a \(T^\omega\)-representation of \(\mathcal{P}(X)\)?
  - No. (Cardinality).
Suppose that $\delta : \subseteq \{0, 1\}^\omega \to X$ is a representation.

- Each infinite sequence (in $\text{dom}(\delta)$) specifies a point.
  - $0101101\ldots \mapsto x \in X$

- Each $T$-sequence specifies an unspecified point.
  - $01\perp 0\perp 1\ldots \mapsto \left\{ \delta(010001\ldots), \delta(010011\ldots), \delta(011001\ldots), \delta(011011\ldots) \right\}$ (If all of them are in $\text{dom}(\delta)$)

- Can we use it as a $T^\omega$-representation of $P(X)$?
  - No. (Cardinality).

- Can we use it as a $T^\omega$-representation of the set $K(X)$ of non-empty compact subsets of $X$?
  - Yes, for $X$ a computably compact computable metric space.
\( \mathbb{T}^\omega \)-sequences as unspecified points.

Suppose that \( \delta : \subseteq \{0, 1\}^\omega \rightarrow X \) is a representation.

- Each infinite sequence (in \( \text{dom}(\delta) \)) specifies a point.
  - \( 0101101... \mapsto x \in X \)

- Each \( \mathbb{T} \)-sequence specifies an unspecified point.
  - \( 01\bot0\bot1... \mapsto \begin{cases} 
\delta(010001...), \\
\delta(010011...), \\
\delta(011001...), \\
\delta(011011...)
\end{cases} \) \( \begin{cases} \text{(If all of them are in } \text{dom}(\delta)) \end{cases} \)

- Can we use it as a \( \mathbb{T}^\omega \)-representation of \( \mathcal{P}(X) \)?
  - No. (Cardinality).

- Can we use it as a \( \mathbb{T}^\omega \)-representation of the set \( \mathcal{K}^-(X) \) of non-empty compact subsets of \( X \)?
  - Yes, for \( X \) a computably compact computable metric space.
Matching representation.

Let $\delta : \subseteq \{0, 1\}^\omega \to X$ and $\psi : \subseteq \mathbb{T}^\omega \to \mathcal{K}^{-}(X)$ be ($\mathbb{T}^\omega$-) representations.

We define as follows. They match if

$$\{ q \mid p \preceq q \in \{0, 1\}^\omega \} \subseteq \text{dom}(\delta) \quad (\forall p \in \text{dom}(\psi))$$

and

$$\psi(p) = \{ \delta(q) \mid p \preceq q \in \{0, 1\}^\omega \}.$$

$\psi$ is hereditary if the restriction of $\psi$ to $\uparrow p$ is a representation of $\mathcal{K}^{-}(\psi(p))$ for $p \in \text{dom}(\psi)$.

$\Rightarrow$ Then, $|A| - 1 \leq \# \{ \bot \in p \}$ for the case $A = \psi(p)$ is a finite set.

$\psi$ is minimal if $|\psi(p)| - 1 = \# \{ \bot \in p \}$ when $|\psi(p)|$ is finite,

$\psi$ is faithful if it is hereditary and minimal.

$\Rightarrow$ If $\psi$ is faithful, then $(\delta, \psi)$ match for $\delta = \psi|_{\{0, 1\}^\omega}$. 

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Let $\delta : \subseteq \{0,1\}^\omega \rightarrow X$ and $\psi : \subseteq T^\omega \rightarrow \mathcal{K}^-(X)$ be ($T^\omega$-)representations. We define as follows.

They match if $\{q \mid p \leq q \in \{0,1\}^\omega\} \subseteq \text{dom}(\delta)$ (\forall p \in \text{dom}(\psi)) and $\psi(p) = \{\delta(q) \mid p \leq q \in \{0,1\}^\omega\}$.

\[\xrightarrow{\omega} X \quad \rightarrow \mathcal{K}^-(X)\]
Matching representation.

Let \( \delta : \subseteq \{0, 1\}^{\omega} \to X \) and \( \psi : \subseteq \mathbb{T}^{\omega} \to K^{-}(X) \) be \((\mathbb{T}^{\omega})\)-representations. We define as follows.

- They match if \( \{ q \mid p \preceq q \in \{0, 1\}^{\omega} \} \subseteq \text{dom}(\delta) \) \((\forall p \in \text{dom}(\psi))\) and \( \psi(p) = \{ \delta(q) \mid p \preceq q \in \{0, 1\}^{\omega} \} \).

- \( \psi \) is hereditary if the restriction of \( \psi \) to \( \uparrow p \) is a representation of \( K^{-}(\psi(p)) \) for \( p \in \text{dom}(\psi) \).

  ▶ Then, \(|A| - 1 \leq \#\{ \bot \in p \} \) for the case \( A = \psi(p) \) is a finite set.

\[ \begin{array}{c}
\bot^{\omega} \\
\downarrow \quad \downarrow \quad \downarrow \\
p \quad \psi(p) \quad K^{-}(X)
\end{array} \]
Matching representation.

Let $\delta : \subseteq \{0, 1\}^\omega \rightarrow X$ and $\psi : \subseteq \mathbb{T}^\omega \rightarrow K^-(X)$ be ($\mathbb{T}^\omega$-)representations. We define as follows.

- They match if $\{ q \mid p \preceq q \in \{0, 1\}^\omega \} \subseteq \text{dom}(\delta)$ ($\forall p \in \text{dom}(\psi)$) and $\psi(p) = \{ \delta(q) \mid p \preceq q \in \{0, 1\}^\omega \}$.

- $\psi$ is hereditary if the restriction of $\psi$ to $\uparrow p$ is a representation of $K^-(\psi(p))$ for $p \in \text{dom}(\psi)$.
  
  - Then, $|A| - 1 \leq \#\{ \bot \in p \}$ for the case $A = \psi(p)$ is a finite set.

- $\psi$ is minimal if $|\psi(p)| - 1 = \#\{ \bot \in p \}$ when $|\psi(p)|$ is finite,

\[
\begin{array}{c}
|A| = 1 \\
|A| = 2 \\
|A| = 3 \\
\vdots
\end{array}
\]

$\bot^\omega$
Matching representation.

Let $\delta : \subseteq \{0, 1\}^\omega \rightarrow X$ and $\psi : \subseteq \mathbb{T}^\omega \rightarrow K^-(X)$ be $(\mathbb{T}^\omega)$-representations. We define as follows.

- They match if $\{ q \mid p \preceq q \in \{0, 1\}^\omega \} \subseteq \text{dom}(\delta)$ $(\forall p \in \text{dom}(\psi))$ and $\psi(p) = \{ \delta(q) \mid p \preceq q \in \{0, 1\}^\omega \}$.
- $\psi$ is hereditary if the restriction of $\psi$ to $\uparrow p$ is a representation of $K^-(\psi(p))$ for $p \in \text{dom}(\psi)$.
  - Then, $|A| - 1 \leq \#\{ \bot \in p \}$ for the case $A = \psi(p)$ is a finite set.
- $\psi$ is minimal if $|\psi(p)| - 1 = \#\{ \bot \in p \}$ when $|\psi(p)|$ is finite,
- $\psi$ is faithful if it is hereditary and minimal.
  - If $\psi$ is faithful, then $(\delta, \psi)$ match for $\delta = \psi|_{\{0,1\}^\omega}$.
Matching representation.

Let $\delta : \subseteq \{0, 1\}^\omega \rightarrow \mathbf{X}$ and $\psi : \subseteq \mathbb{T}^\omega \rightarrow \mathcal{K}^- (\mathbf{X})$ be ($\mathbb{T}^\omega$-)representations. We define as follows.

- They match if $\{q \mid p \preceq q \in \{0, 1\}^\omega\} \subseteq \text{dom}(\delta) \ (\forall p \in \text{dom}(\psi))$ and $\psi(p) = \{\delta(q) \mid p \preceq q \in \{0, 1\}^\omega\}$.

- $\psi$ is hereditary if the restriction of $\psi$ to $\uparrow p$ is a representation of $\mathcal{K}^- (\psi(p))$ for $p \in \text{dom}(\psi)$.
  
  - Then, $|A| - 1 \leq \#\{\bot \in p\}$ for the case $A = \psi(p)$ is a finite set.

- $\psi$ is minimal if $|\psi(p)| - 1 = \#\{\bot \in p\}$ when $|\psi(p)|$ is finite,

- $\psi$ is faithful if it is hereditary and minimal.
  
  - If $\psi$ is faithful, then $(\delta, \psi)$ match for $\delta = \psi|_{\{0,1\}^\omega}$.

**Theorem (Main Theorem)**

*If $\mathbf{X}$ is a CCCMS (computably compact computable metric space), then $\mathcal{K}^- (\mathbf{X})$ admits a faithful $\mathbb{T}^\omega$-representation.*

**Note1:** Applications in finite closed choice and Weihrauch reducibility.

**Note2:** Compact $=$ closed in a compact metric space. $\mathcal{K}(\mathbf{X}) \Leftrightarrow \mathcal{A}(\mathbf{X})$. 

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$\mathbb{T}^\omega$-representations of compact sets through $\omega$ 

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Outline of the proof

- Computably compact computable metric space
- Existence of proper computable dyadic subbases
- Domain representation
- Compact sets are tree-like subsets
  - (Expanding tree-like sets as trees)
- Cantor space (Compact sets are binary trees)
- Faithful representation
Outline of the proof

Computably compact computable metric space
⇓
Existence of proper computable dyadic subbases
⇓
Domain representation
⇓
Compact sets are tree-like subsets
(Expanding tree-like sets as trees)

Cantor space (Compact sets are binary trees)
⇓
Faithful representation
Closed subsets of Cantor space as binary trees.

- A binary tree ($T$) is a prefix-closed subset of $\{0, 1\}^*$. 
- Infinite paths of $T \in \mathcal{T}$ form a closed subset $[T]$ of $\{0, 1\}^\omega$. 

Diagram:

```
  ...0 ...1 ...0 ...1 0 1 00 01 0 1 0 1 0 1 1 0 1 1
    |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
  0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
      |   |   |   |   |   |   |   |
    0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
        |   |   |   |   |   |   |   |
          0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
            |   |   |   |   |   |   |   |
                0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
                  |   |   |   |   |   |   |   |
                      0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
                        |   |   |   |   |   |   |   |
                            0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
                                      |   |   |   |   |   |   |   |
                                              0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

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$T^\omega$-representations of compact sets through dyadic subbases

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Closed subsets of Cantor space as binary trees.

- A binary tree $(T)$ is a prefix-closed subset of $\{0, 1\}^\ast$.
- Infinite paths of $T \in T$ form a closed subset $\{T\}$ of $\{0, 1\}^\omega$.
- A pruned tree $(\mathcal{PT})$ is a tree $T$ such that $w \in T$ implies $\exists v \in T \ w \sqsubseteq v$.
- $[\cdot]$ is a one to one correspondence between $\mathcal{PT}$ and $\mathcal{A}(\{0, 1\}^\omega)$ (Closed subsets of $\{0, 1\}^\omega$).
- Consider $T^\omega$-representation of pruned trees.

\[
\begin{array}{cccccccc}
000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 1 & & & & & & \\
\end{array}
\]
$\mathbb{T}^\omega$-representation $\delta_{\mathcal{PT}} : \mathbb{T}^\omega \rightarrow \mathcal{PT}$ of Pruned Trees

Let $n(p)$ for $p \in \{0, 1\}^*$ be the enumeration of $\{0, 1\}^*$ as blue.

For a pruned tree $T$,

$$\delta_{\mathcal{PT}}^{-1}(T)(n(p)) = \begin{cases} \bot & \text{(both 0}p, 1p \in T) \\ 0 & \text{(0}p \in T) \\ 1 & \text{(1}p \in T) \\ 0 \text{ or 1} & \text{(p} \notin T \text{ (causes redundancy))} \end{cases}$$

Neither do not happen for a pruned tree.

### Domain Diagram

```
  00 01 10 11
  \\/ \/ \/ \\/
  00 01 10 11
  \  \  \  \  
  3  4  5  6
  \  \  \  \  
  0  1  2  3
  \  \  \  
  $\bot$ 0 1 2
```

### Table

<table>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bot$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
</tr>
</tbody>
</table>

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$\mathbb{T}^\omega$-representations of compact sets through dyadic subbases

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Computing $\delta_{PT}$-name incrementally

Let $A$ be the tree $\{0, 1\}^* \setminus (\uparrow p_1 \cup \uparrow p_1 \cup \ldots)$. Given a sequence $p_1, p_2, \ldots \in \{0, 1\}^*$ (as a $\{0, 1\}^*$-name of a closed set) compute a $\delta_{PT}$-name of \textit{Prune}(A).

$$\cdots 0 \ldots 1 \ldots 0 \ldots 1 \ldots 0 \ldots 1 \ldots 0 \ldots 1 \ldots 0 \ldots 1 \ldots 0 \ldots 1$$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

000 001 010 011 100 101 110 111

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

0 1 4 5 10 11 6

$\epsilon$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 $\ldots$
Computing $\delta_{PT}$-name incrementally

Let $A$ be the tree $\{0, 1\}^* \setminus (\uparrow p_1 \cup \uparrow p_1 \cup \ldots)$. Given a sequence $p_1, p_2, \ldots \in \{0, 1\}^*$ (as a $\{0, 1\}^*$-name of a closed set) compute a $\delta_{PT}$-name of $\text{Prune}(A)$.

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 10 & 010 & 011 & 100 & 101 & 110 & 111 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
3 & 4 & 5 & 1 & 2 & \color{red}{1} & \color{red}{6} & \color{red}{11}
\end{array}
\]

Detected $\uparrow 11 \notin A$. 

\[
\begin{array}{cccccccccccccccc}
\bot & \bot & 0 & \bot & \bot & 0 & \bot & \bot & \bot & \bot & 0 & 0 & \ldots \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14
\end{array}
\]
Computing $\delta_{PT}$-name incrementally

Let $A$ be the tree $\{0, 1\}^* \setminus (\uparrow p_1 \cup \uparrow p_1 \cup \ldots)$. Given a sequence $p_1, p_2, \ldots \in \{0, 1\}^*$ (as a $\{0, 1\}^*$-name of a closed set) compute a $\delta_{PT}$-name of $\text{Prune}(A)$.

\[
\begin{array}{ccccccccccccccc}
\ldots & 0 & \ldots & 1 & \ldots & 0 & \ldots & 1 & \ldots & 0 & \ldots & 1 & \ldots & 0 & \ldots & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
00 & 01 & 10 & 11 \\
\end{array}
\]

$0 \uparrow 0 \downarrow 1 \uparrow 1 \downarrow \epsilon$

$\epsilon \downarrow 0 \uparrow 1 \downarrow 2 \uparrow 3 \downarrow 4 \uparrow 5 \downarrow 6 \uparrow 7 \downarrow 8 \uparrow 9 \downarrow 10 \uparrow 11 \downarrow 12 \uparrow 13 \downarrow 14 \ldots$

1. Detected $\uparrow 11 \not\in A$.
2. Detected $\uparrow 100 \not\in A$. 

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$T^\omega$-representations of compact sets through dyadic subbases

Domains 2018 11 / 26
Computing $\delta_{PT}$-name incrementally

Let $A$ be the tree $\{0, 1\}^* \setminus (\uparrow p_1 \cup \uparrow p_2 \cup \ldots)$. Given a sequence $p_1, p_2, \ldots \in \{0, 1\}^*$ (as a $\{0, 1\}^*$-name of a closed set) compute a $\delta_{PT}$-name of $\text{Prune}(A)$.

...0 ...1 ...0 ...0 ...1 ...0 ...1 ...0 ...1 00...1 010 011 100 101 110 111

00 01 4

0 1

$\epsilon$

Detected $\uparrow 11 \notin A$.

Detected $\uparrow 100 \notin A$.

Detected $\uparrow 101 \notin A$. 

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
The number of bottoms

- Let $\mathcal{PT}$ be the $\mathbb{T}^\omega$-represented space $(\mathcal{PT}, \delta_{\mathcal{PT}})$.
- $\delta_T : (\{0, 1\}^*)^\omega \rightarrow \mathbb{T}$, $p_1, p_2, \ldots \mapsto \{0, 1\}^* \setminus (\uparrow p_1 \cup \uparrow p_2 \cup \ldots)$ is a $\{0, 1\}^\omega$-representation. $T = (T, \delta_T)$.
- The above computation shows that $\text{Prune} : T \rightarrow \mathcal{PT}$ is computable.
- Similarly, one can show that its multi-valued inverse is computable.
- $[\cdot] : T \rightarrow \mathcal{A}(\{0, 1\}^\omega)$ is computable with computable multi-valued inverse.
- $\text{Prune} \circ [\cdot]^{-1}$ is a computable isomorphism between $\mathcal{K}(\{0, 1\}^\omega) = \mathcal{A}(\{0, 1\}^\omega)$ and $\mathcal{PT}$, and obtain a $\mathbb{T}^\omega$-representation of $\mathcal{K}(\{0, 1\}^\omega)$.
- It is a hereditary representation.
- If $A$ is a finite set of cardinarity $n$, only $n - 1$ number of $\perp$ is left (minimal).

**Theorem**

$\mathcal{K}^{-1}(\{0, 1\}^\omega)$ has a faithful $\mathbb{T}^\omega$-representation.
Outline of the proof

Computably compact computable metric space
⇓
Existence of proper computable dyadic subbases
⇓
Domain representation
⇓
Compact sets are tree-like subsets
  (Expanding tree-like sets as trees)
Cantor space (Compact sets are binary trees)
⇓
Faithful representation
Unit Interval as Retract of the limit-set of a DAG

- Signed digit representation of the unit interval.
- Three-branching, two-merging DAG (Directed Acyclic Graph).
- Infinite paths (obtained as the ideal completion) are representing the unit interval.
- Note: three essentially different paths represent 0.
Unit Interval as Retract of the limit-set of a DAG

- Signed digit representation of the unit interval.
- Three-branching, two-merging DAG (Directed Acyclic Graph).
- Infinite paths (obtained as the ideal completion) are representing the unit interval.
- Note: three essentially different paths represent 0.
- Tree-like subsets ($\forall p \neq \epsilon \in A (\exists q \in A q \preceq^1 p)$) of this DAG are representing closed subsets.

\[-1, 1\]
Expansion of a DAG to a tree

Question: Which space can be represented as the limit-set of a Scott domain that is obtained as the ideal completion of this kind of DAG?
Outline of the proof

Computably compact computable metric space

⇓

Existence of proper computable dyadic subbases

⇓

Domain representation

⇓

Compact sets are tree-like subsets

(Expanding tree-like sets as trees)

Cantor space (Compact sets are binary trees)

⇓

Faithful representation

\( G : [-1, 1] \rightarrow \mathbb{T}^\omega \): Injective \( \mathbb{T}^\omega \)-representation of the unit interval.

\[
\begin{array}{cccc}
-1 & 0 & 1/3 & 1/2 & 1 \\
-1/4, 1/4 & [0, 1/2] & \\
-1/2, 1/2 & \\
-1, 0 & & & \\
[-1, 1] & \downarrow 10^\omega & 1010^\omega 1110^\omega & \\
010^\omega 110^\omega & 1^\omega 1\downarrow 10^\omega & 10^\omega & \\
00 & 0 \downarrow 1 & 01 & \downarrow 10 & 11 & 1\downarrow 1 & 10 & \\
0 & \downarrow 1 & & & & & & \\
\varepsilon & & & & & & & \\
\end{array}
\]
How is it constructed?

\[ S_{n,0} = \{ x \mid G(x)(n) = 0 \}, \quad S_{n,1} = \{ x \mid G(x)(n) = 1 \}, \]

\[
\begin{align*}
S_{3,0} & = \{ x \mid G(x)(n) = 0 \}, \\
S_{3,1} & = \{ x \mid G(x)(n) = 1 \}, \\
S_{2,0} & = \{ x \mid G(x)(n) = 0 \}, \\
S_{2,1} & = \{ x \mid G(x)(n) = 1 \}, \\
S_{1,0} & = \{ x \mid G(x)(n) = 0 \}, \\
S_{1,1} & = \{ x \mid G(x)(n) = 1 \}, \\
S_{0,0} & = \{ x \mid G(x)(n) = 0 \}, \\
S_{0,1} & = \{ x \mid G(x)(n) = 1 \},
\end{align*}
\]
A dyadic subbase over a set $X$ is a map $S : \mathbb{N} \times \{0, 1\} \to \mathcal{P}(X)$ such that $S_{n,0} \cap S_{n,1} = \emptyset$ for every $n \in \mathbb{N}$ and
$\{(n, i) \mid x \in S_{n,i}\} = \{(n, i) \mid y \in S_{n,i}\} \Rightarrow x = y$ for $x, y \in X$.

$S_{n,\bot} = X \setminus (S_{n,0} \cup S_{n,1})$.

$\varphi_S(x)(n) = \begin{cases} 0 & (x \in S_{n,0}), \\ 1 & (x \in S_{n,1}), \\ \bot & (x \in S_{n,\bot}). \end{cases}$

$\varphi_S : X \to T^\omega :$ embedding into $T^\omega$.

$X_S = (X, \varphi_S^{-1})$ is an admissible $T^\omega$-represented space.

We say that $S$ is a computable dyadic subbase of a represented space $X$ if $X_S$ is computably isomorphic to $X$. 

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Arno Pauly and Hideki Tsuiki

$T^\omega$-representations of compact sets through $\mathbb{D}_{T^\omega}$

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Two kinds of information.

- Each finite sequence $p \in \mathbb{T}^*$ specifies

\[
S(p) = \bigcap_{n \in \text{dom}(p)} S_{n,p(n)}
\]

\[
\bar{S}(p) = \bigcap_{n \in \text{dom}(p)} (X \setminus S_{n,1-p(n)}) = \bigcap_{n \in \text{dom}(p)} (S_{n,p(n)} \cup S_{n,\perp})
\]

Definition

A dyadic subbase $S$ is **proper** if $\bar{S}(p) = \text{cl} S(p)$ for every $p \in \mathbb{T}^*$. 
Recall that $S$ is proper means $\bar{S}(p) = \text{cl} S(p)$ for every $p \in T^*$. 

- Gray-code is a proper dyadic subbase.
- $S_{n,0}$ and $S_{n,1}$ are exteriors of each other. (The case $p = \bot^n 1$.) $S_{n,\bot}$ is the common boundary.

- $S_{0,\bot}$ and $S_{1,\bot}$ do not touch. (The case $p = 00$.)
Computability notions of $\bar{S}(p)$ and $\text{cl } S(p)$

Two computability notions $\mathcal{A}(X)$ (closed) and $\mathcal{V}(X)$ (overt) for closed sets.

- $\bar{S}(p) \in \mathcal{A}(X)$ because $\bar{S}(p) = X \setminus (\bigcup_{n \in \text{dom}(p)} S_{n,1-p(n)})$.
  - $A \in \mathcal{A}(X) \iff A^C \in \mathcal{O}(X)$.
  - Representation by negative information.

- $\text{cl } S(p) \in \mathcal{V}(X)$ because $\text{cl } S(p) = \text{cl } (\bigcap_{n \in \text{dom}(p)} S_{n,p(n)})$.
  - $A \in \mathcal{V}(X)$ is represented by enumeration of $\{U \mid U \cap A \neq \emptyset\}$.
  - Representation by positive information.

- If $S$ is a proper dyadic subbase, then $\bar{S}(p) = \text{cl } S(p) \in \mathcal{V}(X) \land \mathcal{K}(X)$.

- $\{\bar{S}(p) \mid p \in \mathbb{T}^*\}$ is a closed base which consists of overt and closed sets.
Let $S$ be a dyadic subbase of a space $X$. Define

\[
\hat{K}_S = \{ p_{\langle \leq n \rangle} \mid \exists x \in X \varphi_S(x) \preceq p, n \in \mathbb{N} \}
\]

\[
\hat{D}_S = \text{the ideal completion of } \hat{K}_S \subseteq \mathbb{T}^\omega, \quad \hat{L}_S = \hat{D}_S \setminus \hat{K}_S.
\]

**Theorem (T, Tsukamoto, 2015)**

**Suppose that $S$ is a proper computable dyadic subbase of a CCCMS $X$. We have something like**

\[
\begin{array}{cccc}
010^\omega & 110^\omega & \downarrow 10^\omega & 1^\omega 1 \downarrow 10^\omega & 10^\omega \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 \bot 1 & 01 & \downarrow 10 & 11 & 1 \downarrow 1 & 10 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \downarrow 1 & \downarrow 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \bot & \bot & \bot \\
\end{array}
\]

$\hat{K}_S$ : Finite branching DAG.  
$\hat{D}_S$ : Scott Domain  
$\min(\hat{L}_S)$ : Homeomorphic to $X$  
$\hat{L}_S \rightarrow X$ : Computable retract

By expanding it into tree, we have a $\mathbb{T}^\omega$-representation of the set $\mathcal{K}^-(X)$. However, it is not computable because $\hat{K}_S$ is not computable.
To obtain a faithful $\mathbb{T}^\omega$-representation of $\mathcal{K}^-(X)$ from a proper computable dyadic subbase, we consider a bigger domain and construct an algorithm to restrict the limits to $\hat{L}_S$ using the fact that $p \in \hat{K}_S$ is semi-decidable. (Omit the details.)

This is a tree-like set representing the set $\{0, 1/3\}$. Therefore, we need to assign a $\mathbb{T}^\omega$ sequence with one $\perp$.

**Theorem**

*If $S$ is a proper computable dyadic subbase of a CCCMS $X$, then $X$ has a faithful $\mathbb{T}^\omega$-representation.*
Outline of the proof

Computably compact computable metric space
⇓
Existence of proper computable dyadic subbases
⇓
Domain representation
⇓
Compact sets are tree-like subsets
(Expanding tree-like sets as trees)
Cantor space (Compact sets are binary trees)
⇓
Faithful representation
Construction of a proper computable dyadic subbase

**Theorem:** Every separable metric space has a proper dyadic subbase.

- Tsukamoto gave an elegant proof in [Tsukamoto 2017]. (In that paper, he also proved that every locally compact separable metric space has a strongly proper dyadic subbase.) We effectized his proof to obtain

**Theorem (main theorem)**

> If $X$ is a CCCMS, then $\mathcal{K}^{-}(X)$ has a faithful $T^\omega$-representation.

Thank you very much.