Frames and Frame Relations

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- Relational reasoning can get at functional behavior (via, for example, approximable maps).
- These permit us to situate frames in larger ambient categories of relations in which constructions arise from the combination of injectivity and relational reasoning.
- In particular, the assembly of a frame comes about as being isomorphic to a sublocale $\mathcal{Q}(L)$ of the frame of all "weakening" relations a given frame.
- We prove this by showing directly that $\mathcal{Q}(L)$ is such a sublocale and has the universal property of the assembly.

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- Frames are precisely the injective (meet) semilattices.
- Simply knowing this does not get us very far in studying frames *qua* frames.
- But semilattice maps between injective semilattices correspond dually to frame relations (defined below).
- So the general study of frames can be approached via the study of them simply as injective semilattices.

First step: Frame Relations

- A semilattice map $h\colon M \to L$ between two frames can be viewed "dually" as the relation $R_h \subseteq L \times M$ defined by

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- Any such relation, called a frame relation, determines a semilattice homomorphism.
- The category $\overline{\text{Frm}}$ of frames and frame relations is opposite to the full subcategory of SL consisting of injective semilattices. [Note: id_L is the order relation on L.]

Frame homomorphisms and sub-objects

- Suppose $R\colon L \looparrowright M$ and $R_* : M \looparrowright L$ are frame relations satisfying

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- Then there is a frame homomorphism $f\colon L \to M$ so that

$$\begin{aligned} x\,R\,y &\iff f(x) \le y & \qquad \text{and} \\ y\,R_*\,x &\iff y \le f(x) \end{aligned}$$

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- Conversely, every frame homomorphism determines an adjoint pair of frame relations.

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3. *S_R is closed under $\bigwedge$ and $\forall a \in L \forall b \in S, a \to b \in S_L$.*
4. *Any $S \subset L$ satisfying (3) [the sublocale conditions] induces an extremal epi from L to S by restricting $\leq_L$ to $L \times S$.*

Frame pre-congruences

The observations above show that the endo frame relations ϕ satisfying

1. $\mathrm{id}_L \subseteq \phi$; and
2. $\phi ; \phi \leq \phi$

correspond exactly to extremal epis from L (sublocales on L). And

$$\mathcal{Q}(L) = \text{reflexive, transitive frame relations on } L$$

ordered by inclusion is clearly a complete semilattice because meet is intersection.

Frame pre-congruences

Lemma

For any frame L, $\mathcal{Q}(L)$ is a sublocale of $\overline{\mathrm{Pos}}(L, L)$ — the completely distributive lattice of all weakening relations.

Proof.

As already noted, $\mathcal{Q}(L)$ is closed under arbitrary intersections. Suppose $R \colon L \looparrowright L$ is a weakening relation and $\phi \in \mathcal{Q}(L)$. The Heyting arrow in $\overline{\mathrm{Pos}}(L, L)$ by given by

$$x(R \to \phi)y \qquad \text{iff} \qquad \forall w, z \in L, w\, R\, z \Rightarrow w \wedge x\; \phi\; y \vee z.$$

So it is easy to check that $(R \to \phi) \in \mathcal{Q}(L)$. $\square$

Special relations

- For $w \in L$, define $\gamma_w, \upsilon_w \in \mathcal{Q}(L)$ by

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- Now $\Gamma \colon L \looparrowright \mathcal{Q}(L)$ defined by
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- Hence Γ is a frame map, and γ_w and υ_w are complements in $\mathcal{Q}(L)$.

Finally

Theorem

For any frame map $R\colon L \nrightarrow M$ if $R_*; R \subseteq \prec_M$ then there is a unique frame map $R^\dagger\colon \mathcal{Q}(L) \nrightarrow M$ so that

$$R = \Gamma; R^\dagger.$$

Proof.

Define $\beta_w \in \mathcal{Q}(M)$ and $\Lambda\colon \mathcal{Q}(M) \nrightarrow M$ by

$$\frac{x \wedge y^* \leq w}{\overline{\overline{x \; \beta_w \; y}}} \qquad \frac{\phi \subseteq \beta_w}{\overline{\overline{\phi \Lambda w}}}$$

Then checking that $R^\dagger = \mathcal{Q}(R); \Lambda$ satisfies the requirements is a simple calculation. $\square$

Closing summary

Viewing frames as the injective semilattices:

- *Frame relations* are the relational counterparts of semilattice homomorphisms
- *Frame maps* are adjoint frame relations, and correspond to frame homomorphisms.
- The *pre-congruences* on a frame are the reflexive and transitive frame relations.
- These form a frame $\mathcal{Q}(L)$ that directly has the universal property of the frame of all congruences on L.