Extending Stone Duality to Relations

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July 2018

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- One would like to have natural duality for compact Hausdorff structures extending familiar dualities on Stone structures.
- Clearly, this will require us to add something to the algebraic side.
- We know what to do in specific cases: Proximity lattices (Smyth, Jung/Sünderhauf), proximity lattices with "negation" (M).

First step: Relations

- Proximity lattices are distributive lattices equipped with particular sorts of relations.

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- To generalize this, we need to understand how relations generally behave under natural dualities.

Relations Three Ways

Spans: Span

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- A 2-morphism from span $X \xleftarrow{p} R \xrightarrow{q} Y$ to $X \xleftarrow{p'} R' \xrightarrow{q'} Y$ is a monotonic function $f\colon R \to R'$ making the obvious triangles commute.

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Relations three ways

Weakening relations: WRel

- For posets X and Y, a weakening relation is a monotonic map $R\colon X^\partial \times Y \to 2$. Equivalently, identifying with the co-kernel $R = \{(x, y) \mid R(x, y) = 1\}$:

$$\frac{x \leq_X x' \qquad x' \; R \; y' \qquad y' \leq_X y}{x \; R \; y}$$

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- Horizontal composition is defined by the usual relation product.
- A 2-morphism between weakening relations is simply comparison point-wise.

How these are related?

Weakening relations, spans and cospans form 2-categories.
The 2 cells are related via the following functors.

- $R \in \mathrm{WRel}$, determines
  - a span graph(R) by restricting projections
  - a cospan collage(R) by taking the least order on $X \uplus Y$ containing $\leq_X$, $\leq_Y$ and R

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- $X \xleftarrow{p} R \xrightarrow{q} Y$ determines
  - a weakening relation $\mathrm{rel}_s(p,q)$ by $(x,y) \in \mathrm{rel}_s(p,q)$ iff $\exists r \in R$, $x \leq p(r)$ and $q(r) \leq y$
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  - a cospan $\mathrm{cocomma}(p,q)$ by taking the cocomma of (p,q).
- $X \xrightarrow{j} C \xleftarrow{k} Y$ determines
  - a weakening relation $\mathrm{rel}_c(j,k)$ by (x,y) iff $j(x) \leq k(y)$
  - a span $\mathrm{comma}(j,k)$ by taking the comma of (j,k).

How are these related?

We have three 2-categories: Span, Cospan and WRel.
We already described the hom categories: Span(X, Y), Cospan(X, Y) and WRel(X, Y).

- Composition of spans is defined by a comma
- Composition of cospans is defined by a cocomma
- Composition of weakening relations is defined by relational product: $R; S(x, y) = \bigvee_{y \in Y} R(x, y) \wedge S(y, z)$.

How are these related?

The constructions rel_s, rel_c, graph, collage, comma, cocomma are 2-functors:

- $\mathrm{rel}_s(X, Y) \dashv \mathrm{graph}(X, Y)$;
- $\mathrm{rel}_s(X, Y) \circ \mathrm{graph}(X, Y) \cong \mathrm{WRel}(X, Y)$
- $\mathrm{rel}_c(X, Y) \dashv \mathrm{collage}(X, Y)$;
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- $\mathrm{cocomma}(X, Y) \cong \mathrm{collage}(X, Y) \circ \mathrm{rel}_s(X, Y)$.
- These facts hold analogously in PoSpace, the category of topological spaces with closed partial orders with respect to continuous monotonic functions.

Extending to algebras and topological structures

Suppose $\mathcal{A}$ is a class of ordered algebras (algebras with a partial order in which operations are monotone).
Let $\overline{\mathcal{A}}$ denote the category of $\mathcal{A}$-algebra spans in $\mathcal{A}$ with weakening poset reducts.

For example, $\overline{\text{DLat}}$ is the category of bounded distributive lattices with morphisms that are relations satisfying:

- $x \leq x' \; R \; y' \leq y$ implies $x \; R \; y$
- $0 \; R \; 0$
- $1 \; R \; 1$
- $x_0 \; R \; y_0$ and $x_1 \;\; y_1$ implies $x_0 \wedge x_1 \; R \; y_0 \wedge y_1$
- $x_0 \; R \; y_0$ and $x_1 \; R \; y_1$ implies $x_0 \vee x_1 \; R \; y_\vee y_1$.

Main point

Theorem

- $\overline{DL}$ *is (dually equivalent to* $\overline{\text{Priestley}}$*.*
- $\overline{Pos}$ *is dually equivalent to* $\overline{Stone(DLat)}$
- $\overline{SLat}$ *is dually equivalent to* $\overline{Stone(SLat)}$*.*

Proof idea:

- A span $X \xleftarrow{p} R \xrightarrow{q} Y$ in any of the categories mentioned here dualizes to $2^X \xrightarrow{2^p} 2^R \xleftarrow{2^q} 2^Y$ in Priestley.

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- But this transfer preserves the weakening property in each case.
- The correspondence of spans and cospans allows the cospan in the dual category to be tranfered into a span.

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Happy Birthday Dana.
Thanks Klaus.