A logical view of complex analytic maps

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Stone Duality

- Classical Stone duality: 
  \( \text{Boolean Algebras}^{op} \cong \text{Stone Spaces} \).

- Johnstone: \textbf{Sober spaces} \( \cong \text{Spatial locals} \).

- Smyth, Abramsky, Vickers:
  Open sets as observational properties of programs.

- Abramsky 1991: \textit{Domain Theory in Logical Form},
  Denotational Semantics \( \cong \text{Geometric Logic} \).
Geometric Logic

- Open sets of a topological space as propositions or semi-decidable properties.

- \( \Omega(X) \) the lattice of open sets of \( X \).

- Open set \( a \in \Omega(X) \) defines a proposition \( P_a \), with axioms:
  
  (I) If \( a \subseteq b \) then \( P_a \vdash P_b \).
  
  (II) If \( S \) is a family of open sets then \( P_{\cup S} \vdash \bigvee_{a \in S} P_a \).
  
  (III) If \( S \) is a finite family of open sets then \( \bigwedge_{a \in S} P_a \vdash P_{\cap S} \).

- \( \land \emptyset = \text{true}, \lor \emptyset = \text{false} \) then \( P_{\emptyset} \vdash \text{false} \) and \( P_X \vdash \text{true} \).

- \( x \in X \) is a model, \( x \models P_a \) iff \( x \in a \).
Predicate transformer

Dijkstra weakest precondition

- $P$ a program.

- Weakest-precondition of $P$ is a function mapping any postcondition $O$ to a precondition $A$.

- $wp(P, O) = A$. 
Dijkstra weakest precondition

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$x \models A \quad P \quad y \models O$
Differentiation in logical form

- Can we represent derivative of an analytic map $f: U \subseteq \mathbb{C} \rightarrow \mathbb{C}$ by Stone duality?

- Equivalently: Given the predicate transformer representing the map, can we represent the derivative of an analytic map as a predicate transformer?
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Generalized Lipschitz constant (Edalat 2008)

- \( f : U \subseteq \mathbb{C} \rightarrow \mathbb{C} \) a continuous function.
- \( b \) a nonempty convex and compact set in \( \mathbb{C} \).
- \( a \) an open subset of \( U \).
- \( f \) has \textit{set valued Lipschitz constant} \( b \) in \( a \) if:
  \[
  \forall x, y \in a, x \neq y. \quad \frac{f(x) - f(y)}{x - y} \in b
  \]
- Allowing \( b \) any compact interval we have all the local differential properties of \( f \).
- \( \delta(b \chi_a) \), the \textit{tie of} \( a \) \textit{with} \( b \) \textit{collection of all function} \( f \) \textit{which have Lipschitz constant} \( b \) \textit{in} \( a \).
Collecting all the local differential properties captured by ties we obtain the \( L \)-derivative:

\[
\mathcal{L}f : U \subseteq \mathbb{C} \rightarrow \mathbb{C}(\mathbb{C})
\]

\[
\mathcal{L}f(x) = \bigcap \{ b \mid f \in \delta(b \chi_a), x \in a \}
\]

\( \mathcal{L}f \) is Scott continuous function.
Way-below relation in lattice of open sets

- In \( (\Omega(\mathbb{C}), \subseteq) \) we have \( O_1 \ll O_2 \) iff \( \overline{O_1} \subseteq O_2 \).

- \( \Box O = \{ C \in \mathbb{C} | C \subseteq O \} \): basic open set for Scott topology for \( \mathbb{C} \), where \( O \subseteq \mathbb{C} \) is open set.

  way-below relation in \( \Omega(\mathbb{C}) \): \( \Box O_1 \ll \Box O_2 \) iff \( O_1 \ll O_2 \).
Localic approximable mapping

We capture $f$ by approximable mapping $A_f$ which satisfies:

$$a A_f b \text{ iff } a \ll f^{-1}(b)$$
Stably locally compact spaces

- $X$ is stably locally compact if:
  1. $\Omega(X)$ is a distributive continuous lattice.
  2. For $O, O_1, O_2 \in \Omega(X)$, $O \ll O_1 \land O_2$ if $O \ll O_1, O_2$.

- $C, C(C)$ are stably locally compact.

- SLC the category of stably locally compact spaces with continuous functions.

- A semi-strong proximity lattice is a lattice with an additional strong ordering satisfies some axioms.

- PL is the category of semi-strong proximity lattices with approximable mappings.
Equivalence of **PL** and **SLC** via functors **A** and **G**

- **A : SLC → PL.**
- **On objects:** $A(X)$ be the basis of $X$ closed under finite intersection.
- **On morphisms:** $A(f) = A_f$. 

**PL** = **SLC** via **A** & **G** (Jung & Sünderhauf 1996).
Equivalence of **PL** and **SLC** via functors **A** and **G**

- **A : SLC → PL.**
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- **G : PL → SLC.**
- **On objects:** \(G(B) = \text{Spec}(B)\), all prime filters of \(B\).
- **On morphisms:** \(G_R(F) = \{b_2 \in B_2 : \exists b_1 \in F, b_1 \mathrel{R} b_2\}\), for \(R : B_1 \to B_2\).
Equivalence of $\textbf{PL}$ and $\textbf{SLC}$ via functors $A$ and $G$

- $A : \textbf{SLC} \to \textbf{PL}$.
  - **On objects**: $A(X)$ be the basis of $X$ closed under finite intersection.
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- $G : \textbf{PL} \to \textbf{SLC}$.
  - **On objects**: $G(B) = \text{Spec}(B)$, all prime filters of $B$.
  - **On morphisms**: $G_R(F) = \{ b_2 \in B_2 : \exists b_1 \in F. b_1 R b_2 \}$, for $R : B_1 \to B_2$.

- $\textbf{PL} \cong \textbf{SLC}$ via $A$ & $G$ (Jung & Sünderhauf 1996).
Logical representation of $\mathbb{C}$ and $\mathbb{C}(\mathbb{C})$

Semi-strong proximity lattice of $\mathbb{C}$

- $B_{\mathbb{C}}^0$ basis of $\mathbb{C}$ consisting rational convex open polytopes.
- $B_{\mathbb{C}}$ consisting of finite join of elements of $B_{\mathbb{C}}^0$.
- $\prec$ is way-below relation in the lattice of open sets of $\mathbb{C}$ restricted to $B_{\mathbb{C}}$.
- $(B_{\mathbb{C}}, \prec)$ is a semi-strong proximity lattice.
- $\text{Spec}(B_{\mathbb{C}}) \cong \mathbb{C}$.

Similarly, $B_{\mathbb{C}}^0(\mathbb{C})$ basis of Scott topology of $\mathbb{C}(\mathbb{C})$ consisting rational convex open polytopes.

$\text{Spec}(B_{\mathbb{C}}^0(\mathbb{C})) \cong \mathbb{C}(\mathbb{C})$. 
Logical representation of $\mathbb{C}$ and $\mathbb{C}(\mathbb{C})$

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Semi-strong proximity lattice of $\mathbb{C}(\mathbb{C})$

- Similarly, $B_0^0$ basis of Scott topology of $\mathbb{C}(\mathbb{C})$ consisting $\ll a$ for $a \in B_0^0$.
- $(B_{\mathbb{C}(\mathbb{C})}, \prec)$ semi-strong proximity lattice generated by $B_0^0$.
- $\text{Spec}(B_{\mathbb{C}(\mathbb{C})}) \cong \mathbb{C}(\mathbb{C})$. 
Knot of approximable mappings

- Knot of approximable mapping dual to tie of a function.
- $R : B_U \rightarrow B_C$ an approximable mapping.
- $O \in B_C^0$, $a \in B_U^0$.
- $a, O \neq 0, 1$, the bottom and top elements of the lattice.
- $R$ has Lipschitzian constant $O$ in $a$, denoted $R \in \Delta(a, O)$ if,

$$\forall a_1 \forall a_2 \in B_U^0. \ a_1, a_2 \prec a \ & \overline{a_1} \cap \overline{a_2} = \emptyset$$

$$\exists a_1' \exists a_2' \in B_C^0. \ a_1 R a_1' \ & \ a_2 R a_2' \ & \ a_1' - a_2' \prec O \cdot (a_1 - a_2).$$
Strong tie: \( f \in \delta_s(b \chi a) \), if \( \exists a'. a \ll_{\Omega(U)} a' \) & \( \exists b'. b \ll_{C(C)} b' \) such that \( f \in \delta(b' \chi a') \).

Strong knot: \( R \in \Delta_s(a, O) \), if \( \exists a'. a \prec a' \) & \( \exists O'. O' \prec O \) such that \( R \in \Delta(a', O') \).
Stone duality of strong ties and strong knots

- **Strong tie:** \( f \in \delta_s(b \chi a) \), if \( \exists a'. \ a \prec_{\Omega(U)} a' \text{ and } \exists b'. \ b \prec_{C(C)} b' \) such that \( f \in \delta(b' \chi a') \).

- **Strong knot:** \( R \in \Delta_s(a, O) \), if \( \exists a'. \ a \prec a' \text{ and } \exists O'. \ O' \prec O \) such that \( R \in \Delta(a', O') \).

**Theorem:**

- \( R \in \Delta_s(a, O) \) iff \( G_R \in \delta_s(\overline{O} \chi a) \).

- \( f \in \delta_s(b \chi a) \) iff \( A_f \in \Delta_s(a, b^\circ) \).
Example

- Consider absolute value function, i.e., $z \mapsto \bar{z} : \mathbb{C} \to \mathbb{C}$.

- $f \in \delta(b \chi_a) \iff D(-1, 1) \subseteq b$.

- Thus, $\mathcal{L} f(z) = D(-1, 1)$. 
Stone duality for analytic functions

- $R \in \Delta^1(U)$ if:
  \[
  \forall a_0 < 1 \\forall \epsilon > 0 \ \exists \delta > 0. \ a < a_0 \ \& \ \text{diam}(a) < \delta \\
  \exists O \in B^0_C. \ \text{diam}(O) < \epsilon \ \& \ R \in \Delta_s(a, O)
  \]

- $\Delta^1(U)$ class of approximable mapping representing $C^1$ functions.

- $f$ is analytic iff $A_f \in \Delta^1(U)$.

- $R \in \Delta^1(U)$ iff $G_R$ is analytic.
Derivative of approximable mappings

- Single-step approximable mapping \( \eta_{(a,O)} : BU \to BC(\mathbb{C}) \) defined by:

\[
c \eta_{(a,O)} \vartriangleleft V \iff \overline{c} \subseteq a, \quad \overline{O} \subseteq V
\]
Derivative of approximable mappings

- **Single-step approximable mapping** $\eta(a, O) : B_U \rightarrow B_C(\mathbb{C})$
  defined by:
  \[ c \eta(a, O) \square V \iff \overline{c} \subseteq a, \overline{O} \subseteq V \]

- Let $R : B_U \rightarrow B_C$ be a Lipschitzian approximable mapping. The **Lipschitzian derivative** of $R$ is defined as
  \[ \text{L}(R) = \sup\{\eta(a, O) : R \in \Delta_s(a, O)\} \]
Derivative of approximable mappings

- **Single-step approximable mapping** \( \eta_{(a,O)} : B_U \to B_{\mathbb{C}}(\mathbb{C}) \) defined by:

\[
c \eta_{(a,O)} \sqsubset V \iff \overline{c} \subseteq a, \ \overline{O} \subseteq V
\]

- Let \( R : B_U \to B_{\mathbb{C}} \) be a Lipschitzian approximable mapping. The **Lipschitzian derivative** of \( R \) is defined as

\[
L(R) = \sup \{ \eta_{(a,O)} : R \in \Delta_s(a, O) \}
\]

- **Stone duality**, \( L(R) = A_{\mathcal{L}G_R} \).

- \( L(A_f) = A_{\mathcal{L}f} \).
Calculus of the Lipschitzian derivative

- \( R_1, R_2 : B_U \to B_C \) approximable mappings. Then:

\[
L(R_1) + L(R_2) \subseteq L(R_1 + R_2)
\]
equality holds if \( R_1 \) or \( R_2 \) be in \( \Delta^1(U) \).

- \( R_1, R_2 : B_U \to B_C \) approximable mappings. Then:

\[
R_1 \cdot L(R_2) + R_2 \cdot L(R_1) \subseteq L(R_1 \cdot R_2)
\]
equality holds if \( R_1 \) or \( R_2 \) be in \( \Delta^1(U) \).

- \( R_1 : B_{U_1} \to B_C \) and \( R_2 : B_{U_2} \to B_C \) approximable mappings, \( U_1 \subseteq \mathbb{C}, U_2 \subseteq \mathbb{C} \) and \( \text{Im}(R_1) \subseteq B_{U_2} \). Then:

\[
(L(R_2) \circ R_1) \cdot L(R_1) \subseteq L(R_2 \circ R_1)
\]
equality holds if \( R_1 \in \Delta^1(U_1) \) or \( R_2 \in \Delta^1(U_2) \).
Conclusion and future work:

- Implementation in Haskell and Coq.
- Validation of Automatic Differentiation.