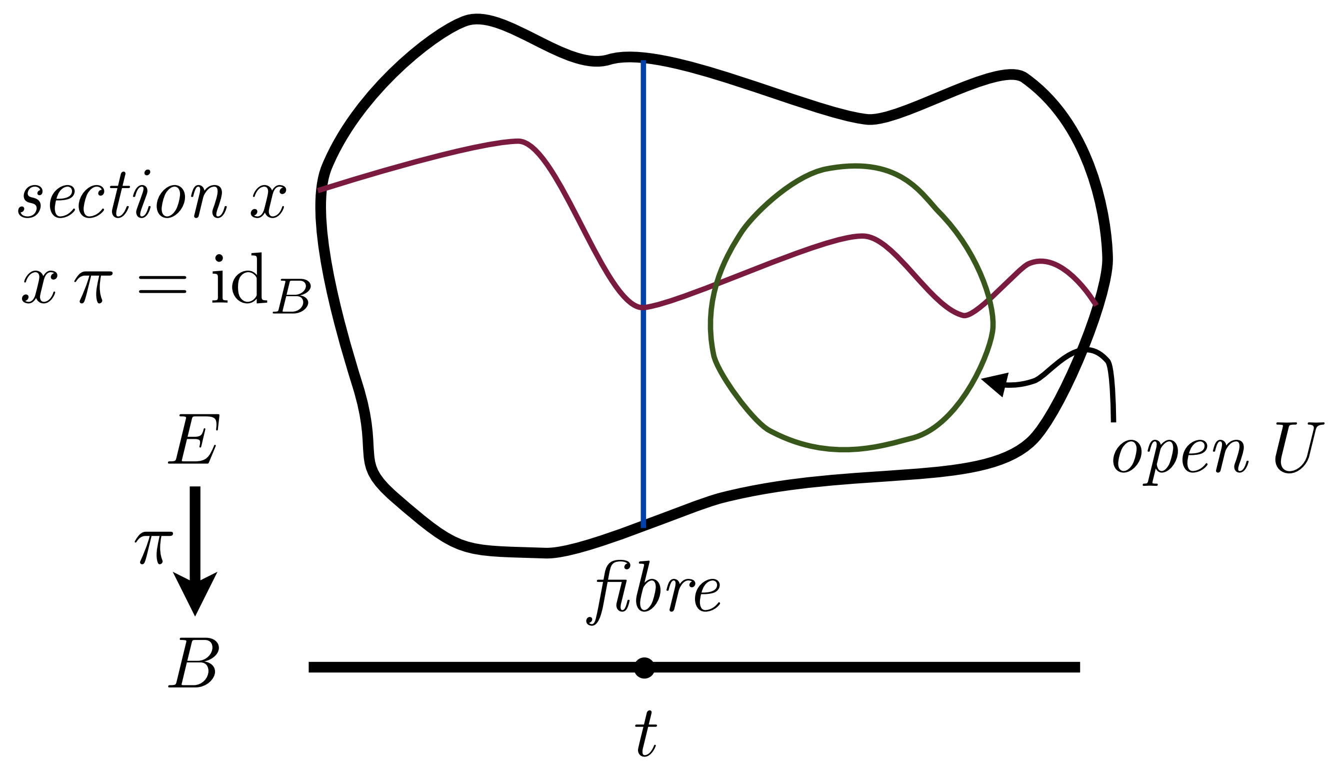
A Brouwerian Proof of the Fan theorem

Michael P. Fourman

dedicated to Dana S. Scott



slides presented at domains13, Oxford 7/7/2018; with minor modifications 9/7/2018

Extending the topological interpretation to intuitionistic analysis

Dedicated to A. Heyting on the occasion of his 70th birthday

by

Dana Scott

Compositio Mathematica, tome 20 (1968), p. 194-210.

The well-known Stone-Tarski interpretation of the intuitionistic propositional logic was extended by Mostowski to the quantifier logic in a natural way. For details and references the reader may consult the work Rasiowa-Sikorski [5], where intuitionistic theories are discussed in general, but where no particular theory is analysed from this point of view. The purpose of this paper is to present some classically interesting models for the intuitionistic theory of the continuum. These models will be applied to some simple independence questions. The idea of the model can also be used for models of second-order intuitionistic arithmetic (cf. the system of [6]), but lack of time and space force us to postpone this discussion to another paper. Also, the author has encountered some difficulty in verifying certain of the continuity assumptions (Axiom F4 of [6] for $\forall\alpha\exists\beta$ to be precise) and hopes to try to understand the motivation behind these principles better before presenting the details of the model. It is not impossible that there are several *distinct* intuitionistic notions of free-choice sequence (real number) with various continuity properties.

Brouwerian:

Act I	twoity	$0, 1$	
	tuples	$\langle a, b \rangle$	
	constructions	integers	
		binary strings	
		finite trees	(finitary inductive definitions)
Act II	species	$a \in U$	(determined by properties)
		choice sequences	(finite prefixes $a \prec \alpha$)
	spreads		(restrictions on free choices)

A *open*, U, of the *formal Cantor space*, $\mathbb{C}$, is a species of finite binary strings $a, b, \ldots \in 2^{<\mathbb{N}}$ that is

$$\text{persistent: } \forall a, b.\, b \geq a \in U \to b \in U$$

$$\text{inductive: } \forall a.\, a\hat{\ }0 \in U \wedge a\hat{\ }1 \in U \to a \in U$$

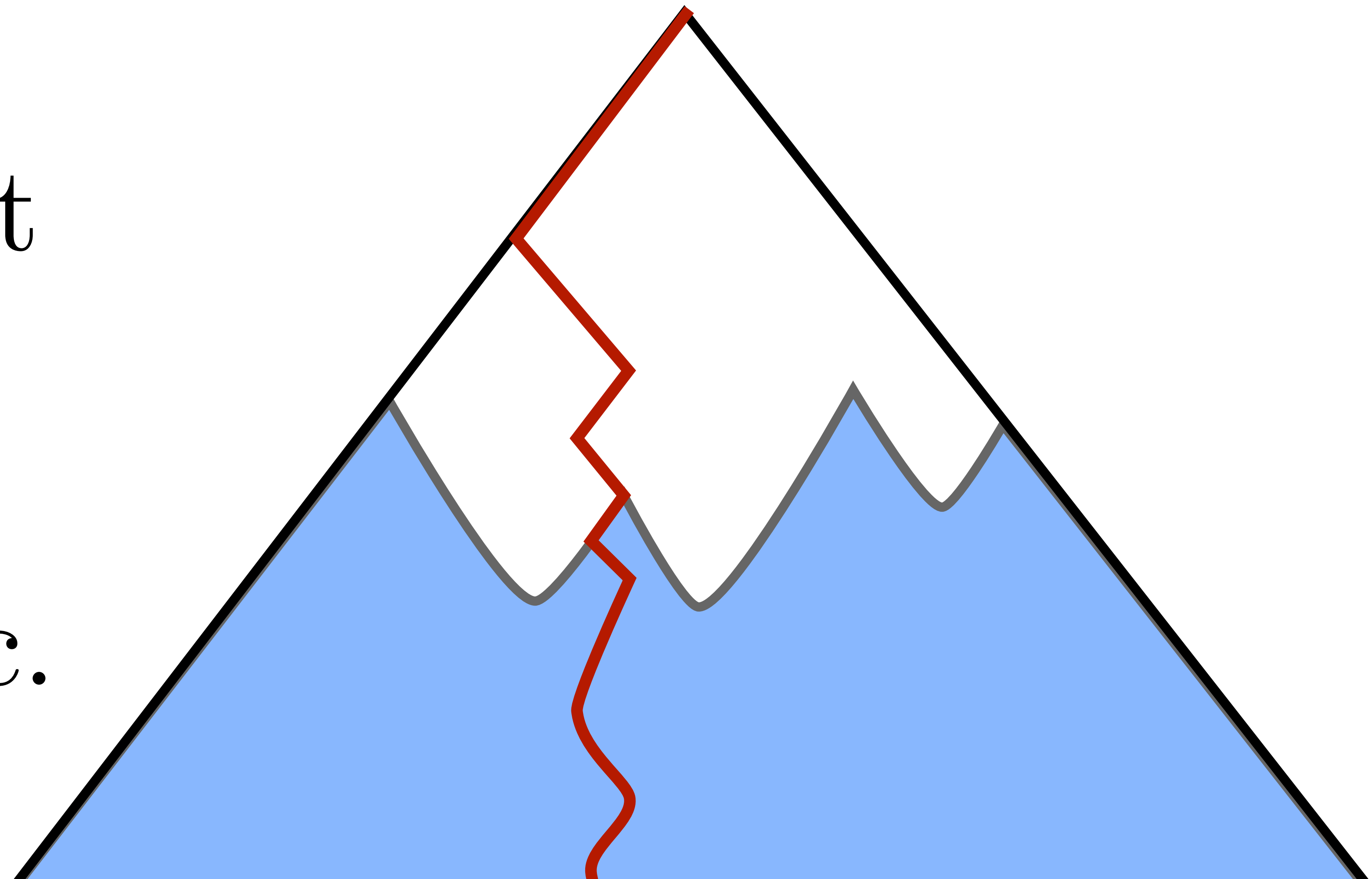
A formal open is a *cover* iff $\forall \alpha : 2^{\mathbb{N}}\, \exists a \in U.\, a \prec \alpha$.

The **Fan Theorem** says,

If U is a formal open cover of $\mathbb{C}$ *then* $\varepsilon \in U$.

The Fan Theorem underpins Brouwer's development of intuitionistic analysis.

It is independent of higher-order Heyting arithmetic.



To justify the Fan Theorem Brouwer introduces a *creating subject*, who can indefinitely extend the stock of mathematical entities by introducing *free choice sequences*.

Choice sequences

A (binary) choice sequence, α, is defined by the species of its finite prefixes.
We write the corresponding property as $a \prec \alpha$.
It must satisfy the following properties:

$$\varepsilon \prec \alpha$$

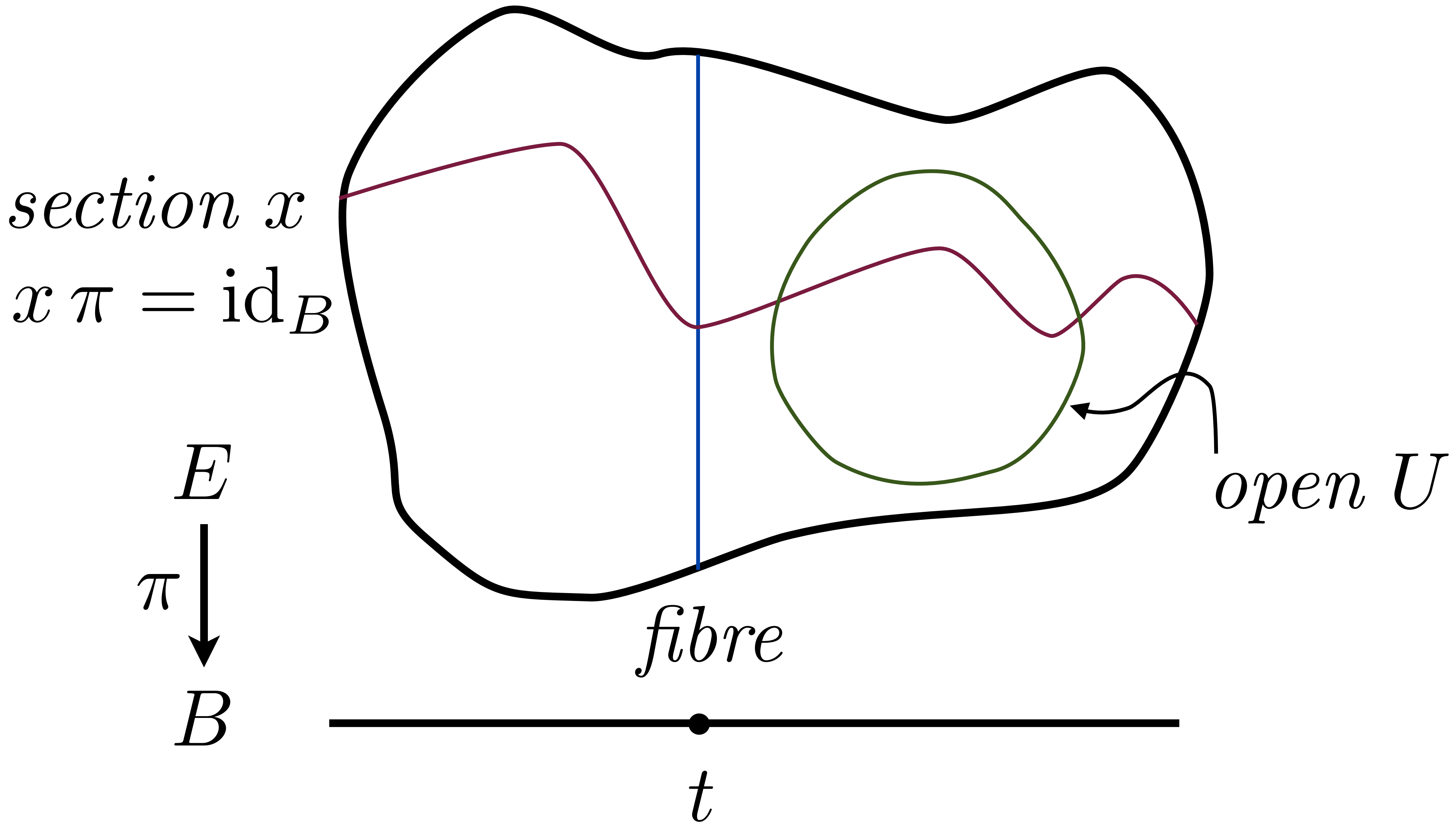
$$a \prec \alpha \leftrightarrow a\hat{\ }0 \prec \alpha \vee a\hat{\ }1 \prec \alpha$$

$$a\hat{\ }0 \prec \alpha \wedge a\hat{\ }1 \prec \alpha \rightarrow \perp$$

At any stage the creating subject can know only a finite initial segment of a free choice sequence; no restrictions are placed on its future values.

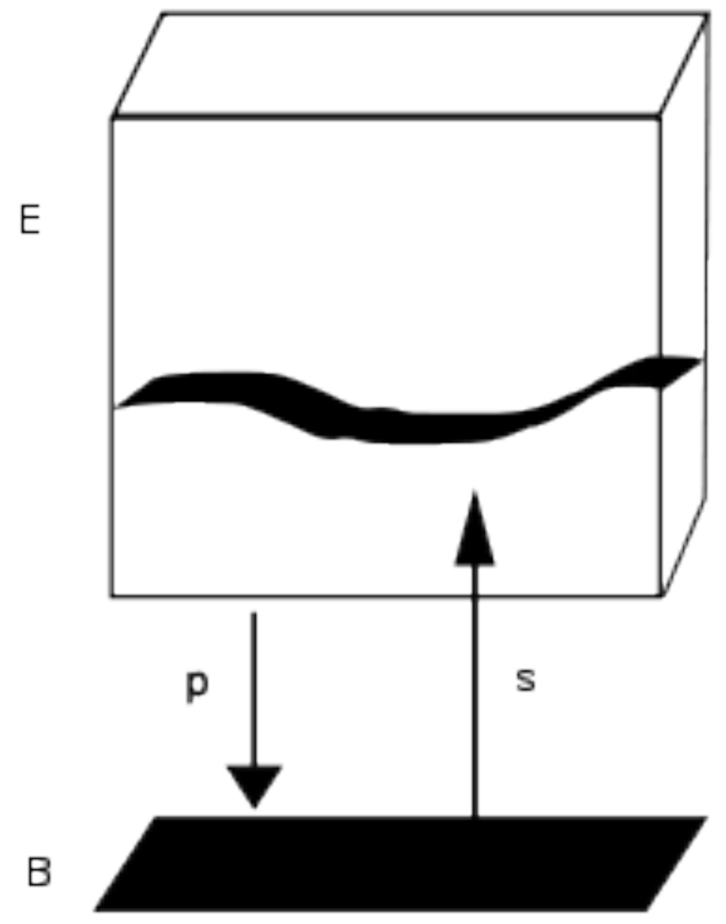
topological interpretation

$[\![\varphi]\!] \in \mathcal{O}(B)$ Truth values $[\![\varphi]\!] \in \mathcal{O}(B)$
in the complete Heyting algebra (cHa) of open sets

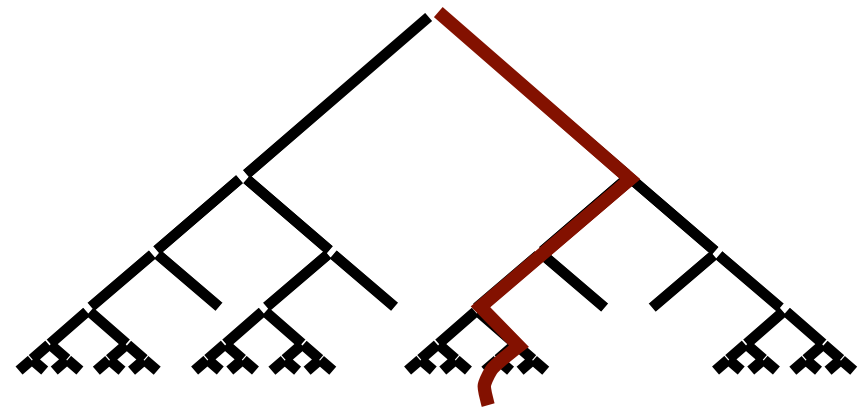


$$[\![x \# y]\!] = \{t \mid x(t) \neq y(t)\}$$

$$[\![x \in U]\!] = x^{-1}(U)$$



The topological interpretation over the formal Cantor space $\mathbb{C}$ corresponds to a Beth model over the tree of finite strings:



$$[\![\varphi]\!] = \{b \mid b \Vdash \varphi\} \qquad \text{(forcing is persistent and inductive)}$$

We interpret this as a model of the activity of a creating subject dependent on a free choice sequence γ:

$$b \Vdash \varphi \text{ iff the information that } b \prec \gamma \text{ justifies the conclusion } \varphi$$

A species U dependent on γ is modelled by giving a truth value

$$[\![a \in U]\!] \in \mathcal{O}(\mathbb{C})\text{for each string } a.$$

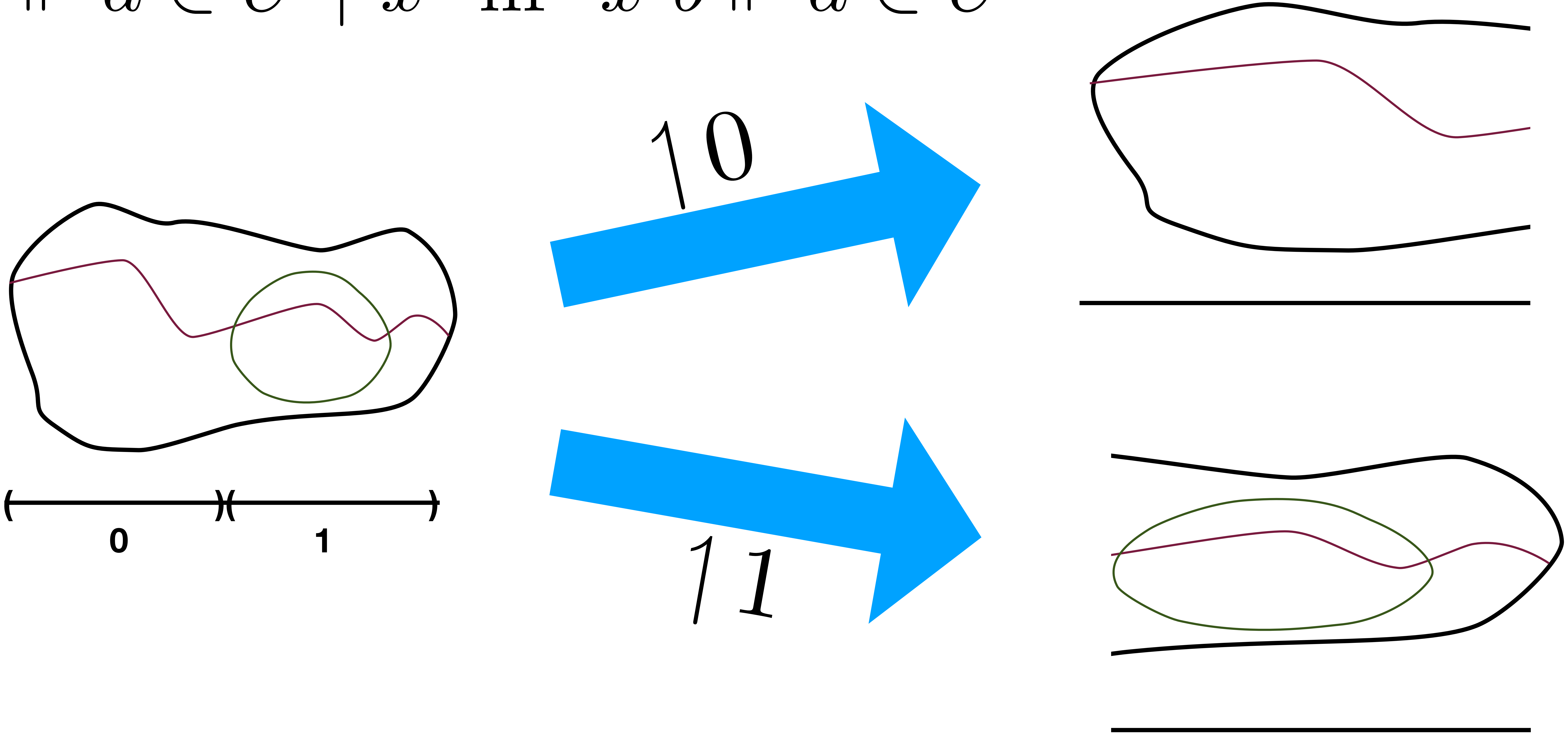
Any choice sequence $a \prec \alpha$ corresponds to a continuous function α with

$$b \Vdash a \prec \alpha \text{ iff } b \in \alpha^{-1}(a)$$

In particular, γ corresponds to the identity function: $b \Vdash a \prec \gamma$ iff $b \geq a$.

We can also view this as a monoid model.
The free monoid on $0, 1$ acts on the topological model.
The monoid action represents a change of perspective of the creating subject.

$$b \Vdash a \in U \restriction x \quad \text{iff} \quad x\,\hat{}\,b \Vdash a \in U$$



Introducing a new free choice sequence.

Two free choice sequences may be interleaved to form a single free choice sequence. We model the introduction of a new choice sequence by transporting our constructions to be the even-indexed subsequence of this interleaving. Then the odd-indexed subsequence represents the new sequence.

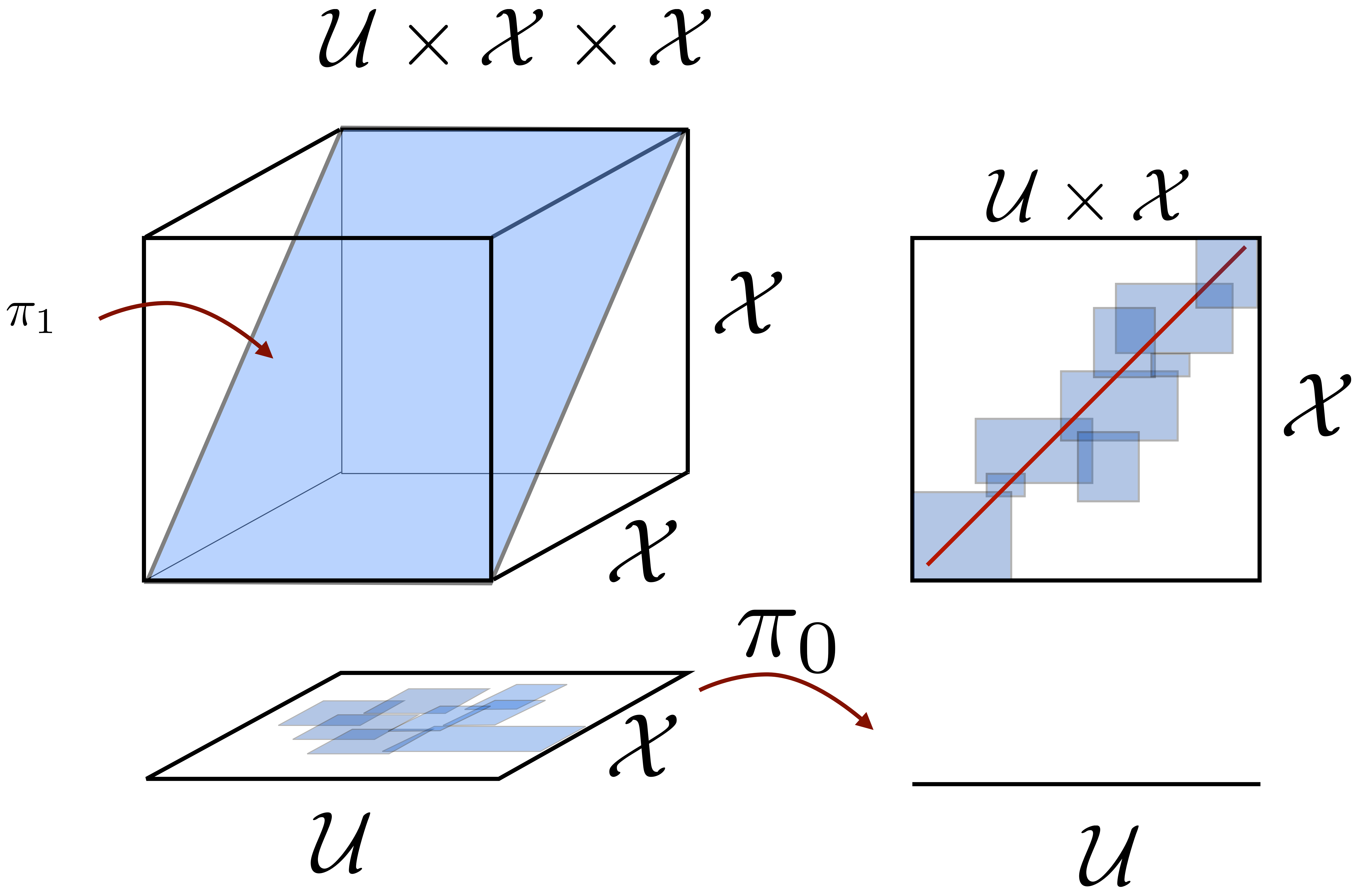
Let b_0 be the even- and b_1 the odd-indexed subsequences of b.

$$b \Vdash a \in V \upharpoonright \pi_0 \text{ iff } b_0 \Vdash a \in V$$

In general, every open map $\mu : \mathbb{C} \longrightarrow \mathbb{C}$ provides a logical endomorphism of the model, with $\alpha \upharpoonright \mu$ given by the composition $\alpha \circ \mu$. A Joyal-Lawvere interpretation of universal quantification brings the new sequence, π_1, in scope:

$$\Vdash \forall \alpha.\, \varphi(\alpha) \text{ iff forall } \mu \in M, \text{ forall } \xi, \Vdash (\varphi \upharpoonright \mu)(\xi).$$

$$\text{In particular, } \Vdash (\varphi \upharpoonright \pi_0)(\pi_1)$$



$\mathcal{U} \times \mathcal{X} \times \mathcal{X}$
π_1
$\mathcal{X}$
$\mathcal{X}$
$\mathcal{U} \times \mathcal{X}$
$\mathcal{X}$
π_0
$\mathcal{X}$
$\mathcal{U}$
$\mathcal{U}$

Lemma A forcing relation $b \Vdash a \in U$ defines a persistent, inductive species iff, viewed externally as a species of pairs of strings, it is persistent and inductive in both a and b.
In this case the species defined by $p_0 \Vdash p_1 \in U$ is persistent and inductive in p.

If U is persistent, inductive, and $\Vdash \forall\alpha\exists a \in U.\, a \prec \alpha$, then $\Vdash \exists a \in U \restriction \pi_0,\ a \prec \pi_1$. The collection of those p such that for some a, both $p_0 \Vdash a \in U$ and $a \leq p_1$, must cover the empty string. Since U is persistent, this is the (by the lemma, inductive) collection of p such that $p_0 \Vdash p_1 \in U$.

Thus the empty string is in this collection, so $\Vdash \varepsilon \in U$.

Instead of spaces use locales (inductive covers).

Instead of open inclusions use open maps.

Instead of sheaves use $\mathcal{O}(\mathbb{C})$-valued models whose elements are constructions – trees that make explicit the stage at which information is available.
Then *define* equality in terms of basic predicates.