Higher-dimensional categories: induction on extensivity

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Leinster’s method for defining weak $n$-categories:

- Start with the free strict $n$-category monad $T^{(n)}$, which is cartesian.
- For a cartesian monad $T$ on $C$, define $T$-operads as monoids in $C/T1$.
- Define a notion of contraction on $T^{(n)}$-operads.
- A weak $n$-category is an algebra for the initial $T^{(n)}$-operad with contraction.

Aims:

- Enrich this to define weak $n$-dimensional $\mathcal{V}$-categories.
- Build dimensions through iterated enrichment.
Extensivity

Definition

A category \( \mathcal{V} \) (with small coproducts) is extensive if, for any set \( I \) and family of objects \((X_i)_{i \in I}\), the functor

\[
\coprod: \prod_{i \in I} (\mathcal{V}/X_i) \rightarrow \mathcal{V}/(\coprod_{i \in I} X_i)
\]

is an equivalence of categories.

Examples: \textbf{Set}, \( \omega\text{-Cpo} \), \textbf{Cat}, \( \mathcal{V}\text{-Cat} \) and \( \mathcal{V}\text{-Gph} \) (for extensive \( \mathcal{V} \)).
Definition

Let $\mathcal{V}$ be a category with finite products. For each natural number $n$, $\mathcal{V}$-$\text{Cat}^{(n)}$ is defined by:

$\mathcal{V}$-$\text{Cat}^{(0)} = \mathcal{V}$; \quad $\mathcal{V}$-$\text{Cat}^{(n+1)} = (\mathcal{V}$-$\text{Cat}^{(n)})$-$\text{Cat}$,

and $\mathcal{V}$-$\text{Gph}^{(n)}$ is defined by:

$\mathcal{V}$-$\text{Gph}^{(0)} = \mathcal{V}$; \quad $\mathcal{V}$-$\text{Gph}^{(n+1)} = (\mathcal{V}$-$\text{Gph}^{(n)})$-$\text{Gph}$.

When $\mathcal{V} = \text{Set}$, $\text{Set}$-$\text{Cat}^{(n)} = n$-$\text{Cat}$, the category of strict $n$-categories.
Proposition

If $\mathcal{V}$ is extensive and finitely complete, then $\mathcal{V}$-$\text{Gph}$ and $\mathcal{V}$-$\text{Cat}$ are also extensive and finitely complete.

Corollary

If $\mathcal{V}$ is extensive and finitely complete, then $\mathcal{V}$-$\text{Gph}^{(n)}$ and $\mathcal{V}$-$\text{Cat}^{(n)}$ are also extensive and finitely complete.
Cartesian monads

**Definition**

A monad \((T, \eta, \mu)\) on \(\mathcal{C}\) is *cartesian* if

- \(\mathcal{C}\) has all pullbacks,
- \(T\) preserves pullbacks,
- all the naturality squares for \(\eta\) and \(\mu\) are pullback squares.

**Proposition**

Let \(\mathcal{V}\) be extensive and finitely complete. For each \(n\), there is an adjunction

\[
\mathcal{V}\text{-Gph}^{(n)} \underoverset{\perp}{\rightarrow}{\leftarrow} \mathcal{V}\text{-Cat}^{(n)}
\]

and the induced monad \(T^{(n)}\) is cartesian.
$T$-operads

For a cartesian monad $T$ on a finitely complete category $C$, $C/T1$ can be given a monoidal structure.

**Definition**

A *$T$-operad* is a monoid in $C/T1$.

Given a $T$-operad $m: M \to T1$, an *algebra for* $(M, m)$ consists of an object $X$ of $C$ together with an action of $(M, m)$ on $X$.

- Classical operads: $C = \textbf{Set}$, $T = \text{free monoid}$.
- For weak $n$-dimensional $\mathcal{V}$-categories, use $C = \mathcal{V}$-$\textbf{Gph}^{(n)}$, $T = T^{(n)}$. 
A *contraction* on \( m: M \to T^{(n)}1 \) consists of a lifting

\[
\begin{array}{ccc}
\partial_j & \xrightarrow{h} & M \\
\downarrow f_j & & \downarrow m \\
C_j & \xrightarrow{k} & T^{(n)}1
\end{array}
\]

for every such commuting square, where \( f_j \) is a “cell boundary inclusion”.

**Definition**

Let \( \mathcal{V} \) be extensive, finitely complete, and locally presentable.

A *weak n-dimensional \( \mathcal{V} \)-category* is an algebra for the initial \( T^{(n)} \)-operad with contraction.

For \( \mathcal{V} = \textbf{Set} \), this agrees with Leinster’s definition of weak \( n \)-category.