Abstractness of Continuation Semantics for Asynchronous Concurrency

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Motivation and Aim

- A **continuation** is a semantic representation of the rest a computation [Stratchey and Wadsworth 1974]

- **Traditional continuations** can express: non-local exits, coroutines, even multitasking and ADA-like rendez-vous

- However, the traditional continuations do not work well enough in the presence of concurrency [Hieb, Dybvig and Anderson 1994]
Motivation and Aim

- In [Todoran 2000, Ciobanu and Todoran 2014] we introduced a **continuation semantics for concurrency (CSC)**

- **CSC** can express **concurrent composition** as well as various **communication and synchronization** mechanisms
  - Intuitively, CSC is a denotational scheduler

- In the CSC approach **continuations** are application-specific structures of computations
  - Rather than the functions to some answer used in the classic technique of continuations

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Motivation and Aim

- In this talk we survey some applications of CSC and we investigate the abstractness of CSC.

- We present an optimality criterion specific of continuation semantics that we name weak abstractness which:
  - Relaxes the completeness condition
  - Preserves the correctness condition of the classic full abstractness criterion [Milner 1977]
## Continuation Semantics for Traditional Concurrent Programming Concepts

<table>
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<tr>
<td>$p ::= (y = x; )^* x$</td>
<td>$p ::= (y = x; )^* x$</td>
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<tr>
<td>$x ::= g</td>
<td>\ll o \gg</td>
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<tr>
<td>$</td>
<td>y</td>
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<td>$g ::= a</td>
<td>\text{fail}$</td>
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<tr>
<td>$l ::= \epsilon</td>
<td>g?x (+g?x)^*$</td>
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<tr>
<td>$o ::= \epsilon</td>
<td>g : x (+g : x)^*$</td>
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<td>$\ll o \gg</td>
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\[ j ::= c?v \mid j\&j \]
\[ a ::= v ::= e \mid c!e \mid j \]
\[ s ::= a \mid y \mid s; s \mid s + s \mid s \parallel s \]

\[ c_1!e_1 \parallel \cdots \parallel c_n!e_n \parallel (c_1?v_1 \& \cdots c_n?v_n) \equiv (v_1, \ldots, v_n := e_1, \ldots, e_n) \]
Continuation Passing Semantics for Nature Inspired Formalisms

Membrane Computing
[Ciobanu and Todoran 2017]

\[ \rho = (D; x), \quad x = o_1 \parallel o_4 \]

\[ D = \text{membrane } M_0 \{ \]
\[ [o_1, o_4] \Rightarrow o_2 \parallel o_4; \]
\[ [o_2] \Rightarrow o_3 \parallel \text{ new}(M_1, l_1, o_1 \parallel o_5); \]
\[ [o_2] \Rightarrow o_4; \]
\[ [o_3] \Rightarrow \text{ in}(l_1, o_5) \parallel o_5; \]
\[ [o_4, o_4, o_5] \Rightarrow o_5; \}; \]
\[ \} \]

\[ \text{membrane } M_1 \{ \]
\[ [o_1] \Rightarrow o_4 \parallel \text{ out}(o_4); \]
\[ [o_5, o_5] \Rightarrow \delta \}. \]

Spiking Neural P-Systems
[Ciobanu and Todoran 2018]

\[ \rho' = (D', x'), \]

\[ x' = \text{ send}(\langle a^{2k-1} \rangle, \{N_1\}) \parallel \text{ send}(a, \{N_3\}) \]

\[ D' = \text{ neuron } N_0 \{ \]
\[ r \epsilon \mid \{N_1, N_2, N_3\} \}
\[ \text{ neuron } N_1 \{ a^+ /[a] \rightarrow a; 2 \mid \{N_2\} \}
\[ \text{ neuron } N_2 \{ [a^k] \rightarrow a; 1 \mid \{N_3\} \}
\[ \text{ neuron } N_3 \{ [a] \rightarrow a; 0 \mid \{N_0\} \}
\]
The full abstractness condition is in general difficult to establish [Milner 1977].

Even more difficult in continuation semantics.

We are not aware of any full abstractness result for a concurrent language designed with continuations.

Continuation-passing semantics for sequential languages are not fully abstract [Cartwright, Curien & Felleisen 1994].
On the Abstractness of Continuation Semantics

- **Weak abstractness** may be useful when full abstractness is difficult (or impossible) to achieve.

- We offer a denotational semantics $\llbracket \cdot \rrbracket$ for an asynchronous formalism; we use a domain of continuations

  $$ D = \text{Cont} \rightarrow \text{P} \quad \text{Cont} = \cdots D \cdots $$

- The semantics is designed by using metric domains [De Bakker and Zucker 1982, America and Rutten 1989]

  - Like the classic domains [Scott 1976, Scott 1982], metric spaces can also be used to express denotational semantics

  - We prove that $\llbracket \cdot \rrbracket$ is weakly abstract w.r.t. an $\mathcal{O}[\cdot]$
Classic Full Abstractness [Milner 1977]

A denotational semantics $D : L \rightarrow D$ is said to be fully abstract with respect to a (corresponding) operational semantics $O : L \rightarrow O$ if

1. $D$ it is correct with respect to $O$
   \[ \forall x_1, x_2 \in L[D(x_1) = D(x_2) \Rightarrow \forall S[O(S(x_1)) = O(S(x_2))]] \]

2. $D$ and complete with respect to $O$
   \[ \forall x_1, x_2 \in L[D(x_1) \neq D(x_2) \Rightarrow \exists S[O(S(x_1)) \neq O(S(x_2))]] \]

$(S$ is an $L$ syntactic context)
Abstractness of Continuation Semantics

- In continuation semantics, \( \mathcal{D} : L \rightarrow \mathcal{D}, \mathcal{D} = \text{Cont} \rightarrow \mathcal{F} \), the completeness condition (of full abstractness) is:

  \[
  \forall x_1, x_2 \in L \left[ (\exists \gamma \in \text{Cont}[\mathcal{D}(x_1) \gamma \neq \mathcal{D}(x_2) \gamma]) \Rightarrow \right.
  
  \left. (\exists S[\mathcal{O}(S(x_1)) \neq \mathcal{O}(S(x_2))]) \right]
  \]

- In continuation semantics for concurrency (CSC) [Todoran 2000, Ciobanu and Todoran 2014] we cannot prove this condition for the whole domain of continuations.
  - A weaker condition can be established if we restrict the investigation to the class of denotable continuations.
    - Which is closed under arbitrary computations.
  - This condition we call weak completeness.
The Class of Metric Domains for CSC

A metric denotational domain of CSC is given as the unique solution of an equation [America and Rutten 1989]

\[ D \cong \Gamma \overset{1}{\rightarrow} F \]

The class \((\Gamma \in)\text{DCONT}\) of domains for CSC is:

\[ \Gamma ::= \frac{1}{2} \cdot D \mid M \rightarrow \Gamma \mid M \times \Gamma \mid M + \Gamma \mid \Gamma \times \Gamma \mid \Gamma + \Gamma \]

\((M\text{ is an arbitrary set }(m \in)M\text{ endowed with the discrete metric})\)
Resumptions and Denotable Continuations

- In the CSC approach a continuation is a structured configuration of (partially evaluated) denotations.

- Following [De Bakker and De Vink 1996] we use the term resumption as an operational counterpart of the term continuation.

- Let \( (x \in) X \) be a fixed set. We define a class \( (R \in) \mathcal{RES}(X) \) of sets of resumptions for \( X \) by:

\[
R ::= X \mid M \rightarrow R \mid M \times R \mid M + R \mid R \times R \mid R + R
\]

(Here \( (m \in) M \) is an arbitrary set.)
Resumptions and Denotable Continuations

- For any language \((x \in)L\) and CSC domain \(\Gamma\) we can construct a corresponding set of resumptions by a homomorphism \(res^L(\cdot) : DCON'T \rightarrow RES(L)\)

\[
res^L(\frac{1}{2} \cdot D) = L
\]
\[
res^L(M \rightarrow \Gamma) = M \rightarrow res^L(\Gamma),
\]
\[
res^L(M \times \Gamma) = M \times res^L(\Gamma)
\]
\[
res^L(M + \Gamma) = M + res^L(\Gamma)
\]
\[
res^L(\Gamma_1 \times \Gamma_2) = res^L(\Gamma_1) \times res^L(\Gamma_2)
\]
\[
res^L(\Gamma_1 + \Gamma_2) = res^L(\Gamma_1) + res^L(\Gamma_2)
\]

- \(res^L(\cdot)\) maps a complete metric space to a plain set
- \(M\) is a metric space, given by an arbitrary set \((m \in)M\) endowed with the discrete metric
Let \((x \in) L\) be a language. Let \(D : L \to D, D \cong \Gamma \to F\), be a denotational semantics of \(L\) designed with CSC.

\((\gamma \in) \Gamma^{Cls} = \{ L_\Gamma | c \in res^L(\Gamma)\}\) is the class of denotable continuations for \(D\), where for any \(\Gamma \in DCON'T\), \(L_\Gamma : res^L(\Gamma) \to \Gamma\) is given by

\[
\begin{align*}
L x_{\Gamma}^2 \cdot D &= D(x) \\
L \lambda m . c_{\Gamma}^L M &\to \Gamma &= \lambda m . L c_{\Gamma}^L \\
L (m, c)_{\Gamma}^L M \times \Gamma &= (m, L c_{\Gamma}^L) \\
L (i, c)_{\Gamma}^L M + \Gamma_2 &= \begin{cases} (1, c) & \text{if } i = 1 \\ (2, L c_{\Gamma}^L) & \text{if } i = 2 \end{cases} \\
L (c_1, c_2)_{\Gamma_1 \times \Gamma_2}^L &= (L c_1_{\Gamma_1}^L, L c_2_{\Gamma_2}^L) \\
L (i, c)_{\Gamma_1 + \Gamma_2}^L &= \begin{cases} (1, L c_{\Gamma_1}^L) & \text{if } i = 1 \\ (2, L c_{\Gamma_2}^L) & \text{if } i = 2 \end{cases}
\end{align*}
\]
The (Metric) Domain of Denotable Continuations

- Let \((\gamma \in \Gamma^{\text{Dom}} = \text{co}(\Gamma^{\text{Cls}} | \Gamma))\) be the metric completion of \(\Gamma^{\text{Cls}}\) with respect to \(\Gamma\) constructed as in the following Remark.
- We call \(\Gamma^{\text{Dom}}\) the domain of denotable continuations for \(\mathcal{D}\).

Remark

Let \((M, d)\) be a complete metric space and let \(X\) be a subset of \(M, X \subseteq M\). We use the notation \(\text{co}(X|M)\) to represent the set

\[
\text{co}(X|M) \overset{\text{not.}}{=} \{ x \mid x \in M, x = \lim_{i} x_i, \forall i \in \mathbb{N} : x_i \in X, (x_i)_i \text{ is a Cauchy sequence in } X \}
\]

where the limits are taken with respect to \(d\) (as \((M, d)\) is complete \(\lim_{i} x_i \in M\)). If we endow \(\text{co}(X|M)\) with \(d_{\text{co}(X|M)} = d\upharpoonright \text{co}(X|M)\) and \(X\) with \(d_X = d\upharpoonright X\), then \((\text{co}(X|M), d_{\text{co}(X|M)})\) is a metric completion of \((X, d_X)\). Recall that each metric space has a completion which is unique up to isometry.
The (Metric) Domain of Denotable Continuations

- We have
  - \( \Gamma^{Cls} \triangleleft \Gamma^{Dom} \), and (by construction) \( \Gamma^{Dom} \triangleleft \Gamma \), but
  - In general, \( \Gamma^{Dom} \neq \Gamma \)

\[ ((M, d) \triangleleft (M', d'), \text{ whenever } M \subseteq M' \text{ and } d' \upharpoonright M = d) \]

- In general we do not know whether \( \Gamma^{Cls} \) itself is a complete metric space (where it is, \( \Gamma^{Cls} = \Gamma^{Dom} \))

- For a simple asynchronous language, in [Ciobanu and Todoran 2012] it is constructed a continuation \( \gamma_\epsilon \in \Gamma \) such that \( d(\gamma, \gamma_\epsilon) \geq \frac{1}{2} \), for any \( \gamma \in \Gamma^{Dom} \)
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Weak Abstractness for CSC

Definition

Let \( (x \in) L \) be a language and let \( D : L \rightarrow D, D \cong \Gamma^1 \rightarrow F, \) be a denotational semantics of \( L \) designed with CSC. Let also \( O : L \rightarrow O \) be an operational semantics of \( L \) and \( S \) a typical element of the class of syntactic contexts for \( L \).

(a) \( D \) is correct with respect to \( O \) iff
\[
\forall x_1, x_2 \in L[D(x_1) = D(x_2) \Rightarrow \forall S[O(S(x_1)) = O(S(x_2))]]
\]

(b) Let \( (\gamma \in) \Gamma^{Dom} \) be the domain of denotable continuations for \( D \). We say that \( D \) is weakly complete with respect to \( O \) iff
\[
\forall x_1, x_2 \in L[(\exists \gamma \in \Gamma^{Dom}[D(x_1)\gamma \neq D(x_2)\gamma]) \Rightarrow
(\exists S[O(S(x_1)) \neq O(S(x_2))])]
\]

(c) We say that \( D \) is weakly abstract with respect to \( O \) iff \( D \) is correct and weakly complete with respect to \( O \).
Weak Abstractness for CSC

- It suffices to verify the weak completeness property for the class of denotable continuations $\Gamma^{Cls}$
- If weak completeness can be established for $\Gamma^{Cls}$ then it holds for the whole domain of denotable continuations $\Gamma^{Dom}$

**Lemma**

Let $(x \in) L$ be a language and let $D : L \rightarrow D, D \cong \Gamma^{1} \rightarrow F$, be a denotational semantics of $L$ designed with CSC. Let also $O : L \rightarrow O$ be an operational semantics of $L$ and $S$ a typical element of the class of syntactic contexts for $L$. $D$ is weakly complete w.r.t. $O$ iff

$$\forall x_1, x_2 \in L[(\exists \gamma \in \Gamma^{Cls}[D(x_1)\gamma \neq D(x_2)\gamma]) \Rightarrow (\exists S[O(S(x_1)) \neq O(S(x_2))])]$$

where $\Gamma^{Cls}$ is the class of denotable continuations for $D$. 

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Abstractness of Continuation Semantics for Asynchronous Concurrency
The Language $\mathcal{L}$ - a Paradigm for Asynchronous Communication [Boer, Kok, Palamidessi, Rutten 1993]

Definition

(a) (Statements) $x \in X ::= a \mid y \mid x + x \mid x ; x \mid x \parallel x \mid x \parallel x$
(b) (Guarded statements) $g \in G ::= a \mid g + g \mid g ; x \mid g \parallel x \mid g \parallel g$
(c) (Declarations, Programs) $(D \in) Decl = Y \rightarrow G, \ (\rho \in) \mathcal{L} = Decl \times X$

- $(a \in) Act$ is a set of atomic actions $(\delta \in Act)$
- $(y \in) Y$ is a set of recursion variables
- $I : Act \rightarrow \Sigma \rightarrow (\{\uparrow\} \cup \Sigma)$ is an interpretation function
  - If $I(a)(\sigma) = \uparrow$ then $a$ is suspended in $\sigma$ $(I(\delta)(\sigma) = \uparrow, \forall \sigma \in \Sigma)$

Instances of the paradigm: asynchronous CCS, asynchronous CSP [Jifeng, Josephs, Hoare 1990], concurrent constraint programming [Saraswat 1993], await statement [Owicki, Gries 1976]
Structure of Continuations (and Resumptions) for $\mathcal{L}$

- $(\alpha \in) A = \{1, 2\}^*$ - identifiers ($\epsilon$ is the empty sequence)
- $\alpha \leq \alpha'$ iff $\alpha' = \alpha \cdot i_1 \cdots i_n$ ($n \geq 0$) - partial order

### Notation for partially ordered bags (multisets) of computations

$$\{|X|\} = \mathcal{P}_{\text{fin}}(A) \times (A \rightarrow X)$$

- A continuation is a **cactus stack** (finite tree, active elements at leaves)
- $\nu : (A \times \{|X|\}) \rightarrow \text{Bool}$ ($\nu(\alpha, (\pi, \theta))$ iff $\alpha$ is a leaf in $(\{\alpha\} \cup \pi, \leq \{\alpha\} \cup \pi)$)

$$\llbracket (x' \parallel x'') ; x \rrbracket \Rightarrow$$

- Cannot define multisets using $(X \rightarrow \mathbb{N})$
Operational Semantics $\mathcal{O}[\cdot]$ 

Semantic universe, configurations, (consistent) resumptions

$$(p \in P) = \mathcal{P}_{nco}(\Sigma^* \cup \Sigma^* \cdot \{\delta\} \cup \Sigma^\omega)$$

$Conf = (X \times CRes' \times \Sigma) \cup (KRes \times \Sigma)$

$CRes = A \times KRes$, $(k \in) KRes = \{\lambda\}, k_0 = (\emptyset, \lambda \alpha. \delta) \in KRes$

$CRes' = \{(\alpha, k) | \alpha \in A, k \in KRes, \nu(\alpha, k)\}$

Operational semantics $\mathcal{O}[\cdot] : X \rightarrow \Sigma \rightarrow P$ \quad ($\mathcal{O} : Conf \rightarrow P$)

$$\mathcal{O}[x](\sigma) = \mathcal{O}(x, (\alpha_0, k_0), \sigma)$$

$$\mathcal{O}(t) = \begin{cases} 
\{\Lambda\} & \text{if } t \text{ terminates} \\
\{\delta\} & \text{if } t \text{ blocks} \\
\cup\{\sigma \cdot \mathcal{O}(k, \sigma) | t \rightarrow (k, \sigma)\} & \text{otherwise}
\end{cases}$$
Transition System Specification for $\mathcal{L}$

\begin{align*}
(a, (\alpha, k), \sigma) & \rightarrow (k, \sigma') \quad \text{if } l(a)(\sigma) = \sigma' \\
(y, (\alpha, k), \sigma) & \uparrow (D(y), (\alpha, k), \sigma) \\
(x_1 + x_2, (\alpha, k), \sigma) & \uparrow (x_1, (\alpha, k), \sigma) \\
(x_1 + x_2, (\alpha, k), \sigma) & \uparrow (x_2, (\alpha, k), \sigma) \\
(x_1; x_2, (\alpha, k), \sigma) & \uparrow (x_1, (\alpha \cdot 1, [k \mid \alpha \mapsto x_2]), \sigma) \\
(x_1 \parallel x_2, (\alpha, k), \sigma) & \uparrow (x_1, (\alpha \cdot 1, [k \mid \alpha \cdot 2 \mapsto x_2]), \sigma) \\
(x_1 \parallel x_2, (\alpha, k), \sigma) & \uparrow (x_1, (\alpha \cdot 1, [k \mid \alpha \cdot 2 \mapsto x_2]), \sigma) \\
(x_1 \parallel x_2, (\alpha, k), \sigma) & \uparrow (x_2, (\alpha \cdot 1, [k \mid \alpha \cdot 2 \mapsto x_1]), \sigma) \\
(k, \sigma) & \uparrow (k(\alpha), (\alpha, k \setminus \{\alpha\}), \sigma) \quad \forall \alpha \in \text{max}(id(k))
\end{align*}
CSC Domains and Evaluation Mechanism

**CSC domains**

\[(\varphi \in D) \iff \text{Cont} \xrightarrow{1} \Sigma \rightarrow P \]

\[(\gamma \in \text{Cont}) = A \times \text{Kont} \]

\[(\kappa \in \text{Kont}) = \{1/2 \cdot D\} \]

**CSC evaluation mechanism**

- **Cont** - open continuations (evaluation contexts)
- **Kont** - closed continuations (manipulated by the scheduler)
- "Evaluate-normalize-schedule" loop
Auxiliary Operators

Non-deterministic choice ($+: (P \times P) \rightarrow P$)

$$p_1 + p_2 = \{q \mid q \in p_1 \cup p_2, q \neq \delta\} \cup \{\delta \mid \delta \in p_1 \cap p_2\}$$

Denotational scheduler

$$kc : Kont \rightarrow \Sigma \rightarrow P$$

$$kc(\kappa)(\sigma) = \begin{cases} \{\Lambda\} & \text{if } (id(\kappa) = \emptyset) \text{ then} \\ +_{\alpha \in \max(id(\kappa))} \kappa(\alpha)(\alpha, \kappa \setminus \{\alpha\})(\sigma) & \text{else} \end{cases}$$

$$\llbracket x \rrbracket(\alpha, \kappa)(\sigma) = c \ p$$ is an abbreviation for:

$$\llbracket x \rrbracket(\alpha, \kappa)(\sigma) = \begin{cases} p & \text{if } \nu(\alpha, \kappa) \\ \{\delta\} & \text{otherwise} \end{cases}$$
Denotational Semantics $\llbracket \cdot \rrbracket : \mathcal{X} \to \mathcal{D}$

$(\mathcal{D}\llbracket \cdot \rrbracket : \mathcal{X} \to \Sigma \to \mathcal{P})$

\[
\begin{align*}
\llbracket a \rrbracket (\alpha, \kappa)(\sigma) &= c & \text{if } (I(a)(\sigma) = \uparrow) \text{ then } \{\delta\} \\
& & \text{else } I(a)(\sigma) \cdot kc((\kappa)(I(a)(\sigma))) \\
\llbracket y \rrbracket (\alpha, \kappa)(\sigma) &= c & \llbracket D(y) \rrbracket (\alpha, \kappa)(\sigma) \\
\llbracket x_1 + x_2 \rrbracket (\alpha, \kappa)(\sigma) &= c & \llbracket x_1 \rrbracket (\alpha, \kappa)(\sigma) + \llbracket x_2 \rrbracket (\alpha, \kappa)(\sigma) \\
\llbracket x_1 ; x_2 \rrbracket (\alpha, \kappa)(\sigma) &= c & \llbracket x_1 \rrbracket (\alpha \cdot 1, [\kappa | \alpha \mapsto \llbracket x_2 \rrbracket ])(\sigma) \\
\llbracket x_1 \parallel x_2 \rrbracket (\alpha, \kappa)(\sigma) &= c & \llbracket x_1 \rrbracket (\alpha \cdot 1, [\kappa | \alpha \cdot 2 \mapsto \llbracket x_2 \rrbracket ])(\sigma) + \\
& & \llbracket x_2 \rrbracket (\alpha \cdot 1, [\kappa | \alpha \cdot 2 \mapsto \llbracket x_1 \rrbracket ])(\sigma)
\end{align*}
\]

$\mathcal{D}[x] = \llbracket x \rrbracket (\alpha_0, \kappa_0), \quad \alpha_0 = \epsilon, \kappa_0 = (\emptyset, \lambda \alpha. [\delta])$
Isomorphic Resumptions

Definition

Two open resumptions \((\alpha_1, k_1), (\alpha_2, k_2) \in CRes\) are isomorphic, written \((\alpha_1, k_1) \cong (\alpha_2, k_2)\), iff either (1) or (2) is satisfied:

(1) \(\neg \nu(\alpha_1, k_1)\) and \(\neg \nu(\alpha_2, k_2)\) ((\(\alpha_1, k_1\) and \(\alpha_2, k_2\) are both inconsistent)

(2) \(\nu(\alpha_1, k_1)\) and \(\nu(\alpha_2, k_2)\) ((\(\alpha_1, k_1\) and \(\alpha_2, k_2\) are both consistent) and there exists a bijection \(\mu : (\{\alpha_1\} \cup id(k_1)) \rightarrow (\{\alpha_2\} \cup id(k_2))\) such that:

(i) \(\mu(\alpha_1) = \alpha_2\)

(ii) \(\mu(\alpha') \leq \mu(\alpha'') \iff \alpha' \leq \alpha'', \; \forall \alpha', \alpha'' \in (\{\alpha_1\} \cup id(k_1))\)

(iii) \(k_2(\mu(\alpha')) = k_1(\alpha'), \; \forall \alpha' \in id(k_1)\)
Class of Denotable Continuations

Definition

We define \( \llbracket \cdot \rrbracket : KRes \rightarrow Kont, \llbracket k \rrbracket = (id(k), \lambda \alpha. \llbracket k(\alpha) \rrbracket) \). The class of (open) denotable continuations for \( \mathcal{L} \) is

\[
Cont^{Cls} = \{(\alpha, \llbracket k \rrbracket) \mid (\alpha, k) \in CRes\} = A \times Kont^{Cls}
\]

where \((\kappa \in)Kont^{Cls} = \{\llbracket k \rrbracket \mid k \in KRes\}\).
Syntactic Contexts for $\mathcal{L}$

**Definition**

*The class of syntactic contexts for $\mathcal{L}$ is given by:*

\[
S ::= (\cdot) \mid a \mid y \mid S; S \mid S + S \mid S\parallel S \mid S\downarrow S
\]

$S(x)$ is the result of substituting $x$ for all occurrences of $(\cdot)$ in $S$.

Let $x, \overline{x} \in X$. When \([ S(x) ](\alpha, \llbracket k \rrbracket) = \llbracket S(\overline{x}) \rrbracket(\overline{\alpha}, \llbracket \overline{k} \rrbracket) \) (*) for all contexts $S$ and for all isomorphic resumptions $(\alpha, k) \cong (\overline{\alpha}, \overline{k}) \in CRes$, we write:

\[x \cong \overline{x}\]

(*) implies $D[S(x)] = D[S(\overline{x})]$, for all contexts $S$.
Concurrence Laws in Continuation Semantics
[Ciobanu and Todoran 2014]

\[
\begin{align*}
x_1 + x_2 & \equiv x_2 + x_1 \\
(x_1 + x_2) + x_3 & \equiv x_1 + (x_2 + x_3) \\
x + x & \equiv x \\
(x_1 + x_2); x_3 & \equiv x_1; x_3 + x_2; x_3 \\
x_1; (x_2 + x_3) & \equiv x_1; x_2 + x_1; x_3 \\
x_1; (x_2; x_3) & \equiv (x_1; x_2); x_3 \\
x + \delta & \equiv x \\
\delta x & \equiv \delta \\
x_1 \parallel x_2 & \equiv x_1 \parallel x_2 + x_2 \parallel x_1 \\
a \parallel x & \equiv a; x \\
a; x_1 \parallel x_2 & \equiv a; (x_1 \parallel x_2) \\
(x_1 + x_2) \parallel x_3 & \equiv x_1 \parallel x_3 + x_2 \parallel x_3
\end{align*}
\]
Concurrency Laws over the Domain of Denotable Continuations

- We do not know whether the class $\text{Cont}^{\text{Cls}}$ of denotable continuations is a complete metric space.

- However, in CSC computations are (nonexpansive and thus) continuous functions, because

$$D \cong \text{Cont}^{1 \rightarrow \Sigma \rightarrow P}$$

- It is reasonable to study semantic properties in the metric completion of the class $\text{Cont}^{\text{Cls}}$, i.e., in the domain of denotable continuations

$$\text{Cont}^{\text{Dom}} = \text{co}(\text{Cont}^{\text{Cls}} | \text{Cont})$$
Concurrency Laws over the Domain of Denotable Continuations

**Definition**

*We say that two open denotable continuations* $(\alpha, \kappa), (\overline{\alpha}, \overline{\kappa}) \in \text{Cont}^{\text{Dom}}$ *are isomorphic, and we write* $(\alpha, \kappa) \cong (\overline{\alpha}, \overline{\kappa})$, *iff there exist sequences* $(\alpha_i, k_i), (\overline{\alpha}_i, \overline{k_i})$, *(with* $(\alpha_i, k_i), (\overline{\alpha}_i, \overline{k_i}) \in \text{CRes}, \forall i \in \mathbb{N})$ *such that:*

- $(\alpha, \kappa) = \lim_i (\alpha_i, [k_i])$, $(\overline{\alpha}, \overline{\kappa}) = \lim_i (\overline{\alpha}_i, \overline{k_i})$, *and*
- $(\alpha_i, k_i) \cong (\overline{\alpha}_i, \overline{k_i}), \forall i \in \mathbb{N}$. 

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Abstractness of Continuation Semantics for Asynchronous Concurrency
Concurrency laws hold for the whole domain $\mathit{Cont}^{\mathit{Dom}}$ of denotable continuations

Let $x, \overline{x} \in X$. When $[S(x)](\alpha, \kappa) = [S(\overline{x})](\overline{\alpha}, \overline{\kappa})$ for all $\mathcal{L}$ syntactic contexts $S$ and for all isomorphic denotable continuations $(\alpha, \kappa) \cong (\overline{\alpha}, \overline{\kappa}) \in \mathit{Cont}^{\mathit{Dom}}$, we write:

$$x \sim \overline{x}$$

**Proposition**

$$x \sim \overline{x} \Rightarrow x \sim \overline{x}, \text{ for all } x, \overline{x} \in X.$$

**Remark**

Not all continuations are denotable: $\mathit{Cont}^{\mathit{Cls}} \triangleleft \mathit{Cont}^{\mathit{Dom}}$ and $\mathit{Cont}^{\mathit{Dom}} \triangleleft \mathit{Cont}$, but $\mathit{Cont}^{\mathit{Dom}} \neq \mathit{Cont}$ [Ciobanu and Todoran 2012]. As a consequence, $\llbracket \cdot \rrbracket$ is not fully abstract.
Recall that \( D \cong Cont \stackrel{1}{\to} (\Sigma \to \mathcal{P}), \) \( Cont = A \times \{ \frac{1}{2 \cdot D} \} \).

If we expand the notation \( \{ \cdot \} \) we get (\( Cont \in DCONT \)):

\[
Cont = A \times (P_{\text{fin}}(A) \times (A \to \frac{1}{2 \cdot D}))
\]

The corresponding set of resumptions is (\( CRes \in RES(X) \))

\[
CRes = res^X(Cont) = A \times (P_{\text{fin}}(A) \times (A \to X))
\]

The class of denotable continuations is

\[
Cont^{Cls} = \{ \llbracket (\alpha, k) \rrbracket^X_{Cont} | (\alpha, k) \in res^X(Cont) \}
\]

\[
= \{ (\alpha, \llbracket k \rrbracket) | (\alpha, k) \in CRes \}.
\]

Lemma

For any \( x \in X, (\alpha, k) \in CRes \) there is an \( L \) syntactic context \( S \) such that:

\[
\llbracket x \rrbracket (\alpha, \llbracket k \rrbracket) = D[ S(x) ] = \llbracket S(x) \rrbracket (\alpha_0, \kappa_0).
\]

Furthermore, \( S \) does not depend on \( x \), it only depends on \( (\alpha, k) \).
The denotational semantics \( [\cdot] \) of \( \mathcal{L} \) is weakly abstract with respect to the operational semantics \( \mathcal{O}[\cdot] \).

Proof.

- One can check that \( \mathcal{D}[x] = \mathcal{O}[x], \forall x \in X \), which implies correctness of \( [\cdot] \) with respect to \( \mathcal{O}[\cdot] \) [Todoran 2000, Ciobanu and Todoran 2012]

- For weak completeness, suppose \( x_1, x_2 \in X, (\alpha, k) \in CRes \) are such that \( [x_1](\alpha, [k]) \neq [x_2](\alpha, [k]) \). By previous Lemma \( \exists S \) such that

\[
\mathcal{D}[S(x_1)] = [x_1](\alpha, [k]) \neq [x_2](\alpha, [k]) = \mathcal{D}[S(x_2)]
\]

- Hence, \( \mathcal{O}[S(x_1)] = \mathcal{D}[S(x_1)] \neq \mathcal{D}[S(x_2)] = \mathcal{O}[S(x_2)] \),

- We conclude that \( [\cdot] \) is (weakly complete and thus) weakly abstract with respect to \( \mathcal{O}[\cdot] \).
Concluding Remarks and Future Research

- We introduce an optimality criterion specific of continuation semantics that we call **weak abstractness**
  - Which relaxes the completeness condition of the classic **full abstractness** criterion [Milner 1977].
- To illustrate the approach we presented a weakly abstract continuation semantics for an asynchronous language
  - We believe similar **weak abstractness** results can be obtained for various **advanced concurrent control concepts**
    - Andorra Model [Todoran and Papaspyrou 2000]
    - Multiparty interactions [Ciobanu and Todoran 2015]
    - Nature inspired formalisms [Ciobanu and Todoran 2017]

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