Approximating partial by total: fixpoint characterizations of back-and-forth equivalences

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Based on:

- The pebbling comonad in finite model theory, SA, Anuj Dawar and Pengming Wang, LiCS 2017
- Relating Structure to Power: comonadic semantics for computational resources, SA and Nihil Shah, to appear in CSL 2018.

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- In most cases of interest in FMT, the converse is true too.
- In descriptive complexity, we seek to characterize a complexity class **C** (for decision problems) as those classes of structures $\mathcal{K}$ (e.g. graphs) definable in $\mathcal{L}$.

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Similarly, there are k-pebble games, and bismulation games played to depth k.

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Just as for EF-games, there is an existential-positive version, in which Spoiler only plays in $\mathcal{A}$, and Duplicator responds in $\mathcal{B}$.

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- Thus the notion of local approximation built into the game is internalised into the category of σ-structures and homomorphisms.
- This leads to comonadic and coalgebraic characterisations of a number of central concepts in Finite Model Theory and combinatorics.

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Our setting will be $\mathcal{R}(\sigma)$, the category of relational structures and homomorphisms.

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This is easily verified to yield a comonad on $\mathcal{R}(\sigma)$.

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Theorem

The following are equivalent:

1. *There is a homomorphism $\mathbb{E}_k \mathcal{A} \to \mathcal{B}$.*
2. *Duplicator has a winning strategy for the existential Ehrenfeucht-Fraïssé game with k rounds, played from $\mathcal{A}$ to $\mathcal{B}$.*
3. *For every existential positive sentence φ with quantifier rank $\leq k$, $\mathcal{A} \models \varphi \;\Rightarrow\; \mathcal{B} \models \varphi$.*

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Given a structure $\mathcal{A}$, the universe of $\mathbb{P}_k\mathcal{A}$ is $(\mathbf{k} \times A)^+$, the set of finite non-empty sequences of moves (p, a). Note this will be infinite even if $\mathcal{A}$ is finite.
We showed that this is essential!

The counit map $\varepsilon_{\mathcal{A}} : \mathbb{E}_k\mathcal{A} \to \mathcal{A}$ sends a sequence $[(p_1, a_1), \ldots, (p_n, a_n)]$ to a_n.

How do we lift the relations on $\mathcal{A}$ to $\mathbb{E}_k\mathcal{A}$?

Given e.g. a binary relation R, we define $R^{\mathbb{P}_k\mathcal{A}}$ to the set of pairs (s, t) such that

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Given a homomorphism $f : \mathbb{P}_k\mathcal{A} \to \mathcal{B}$, we define the coextension $f^* : \mathbb{P}_k\mathcal{A} \to \mathbb{P}_k\mathcal{B}$ by

$$f^*[(p_1, a_1), \ldots, (p_j, a_j)] = [(p_1, b_1), \ldots, (p_j, b_j)],$$

where $b_i = f[(p_1, a_1), \ldots, (p_i, a_i)]$, $1 \leq i \leq j$.

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The universe of $\mathbb{M}_k(\mathcal{A}, a)$ comprises $[a]$, which is the distinguished element, together with all sequences of the form $[a_0, \alpha_1, a_1, \ldots, \alpha_j, a_j]$, where $a = a_0$, $1 \leq j \leq k$, and $R^{\mathcal{A}}_{\alpha_i}(a_i, a_{i+1})$, $0 \leq i < j$.

Simulation

The resource index of $\mathbb{M}_k$ corresponds to the *level of approximation* in simulation $\preceq_k$ and bisimulation $\sim_k$.

Theorem

Let $\mathcal{A}$, $\mathcal{B}$ be Kripke structures, with $a \in A$ and $b \in B$, and $k > 0$. The following are equivalent:

1. *There is a homomorphism $f : \mathbb{M}_k(\mathcal{A}, a) \to (\mathcal{B}, b)$.*
2. *$a \preceq_k b$.*
3. *There is a winning strategy for Duplicator in the k-round simulation game from $(\mathcal{A}, a)$ to $(\mathcal{B}, b)$.*

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In each case, we write

- $\exists\mathcal{L}_k$ for the existential positive fragment of $\mathcal{L}_k$
- $\mathcal{L}_k(\#)$ for the extension of $\mathcal{L}_k$ with counting quantifiers $\exists_{\leq n}$, $\exists_{\geq n}$

Characterization

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We can generically define two equivalences based on our indexed comonads $\mathbb{E}_k$:

- $\mathcal{A} \rightleftarrows_k^{\mathbb{E}} \mathcal{B}$ iff there are coKleisli morphisms $\mathbb{E}_k\mathcal{A} \to \mathcal{B}$ and $\mathbb{E}_k\mathcal{B} \to \mathcal{A}$. Note that there need be no relationship between these morphisms.
- $\mathcal{A} \cong_k^{\mathbb{E}} \mathcal{B}$ iff $\mathcal{A}$ and $\mathcal{B}$ are isomorphic in the coKleisli category $\mathsf{Kl}(\mathbb{E}_k)$.

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Theorem

For structures $\mathcal{A}$ and $\mathcal{B}$:

$$\begin{array}{lcl} \mathcal{A} \equiv^{\exists\mathcal{L}_k} \mathcal{B} & \Longleftrightarrow & \mathcal{A} \rightleftarrows_k \mathcal{B}. \\ \mathcal{A} \equiv^{\mathcal{L}_k(\#)} \mathcal{B} & \Longleftrightarrow & \mathcal{A} \cong_{\mathsf{Kl}(\mathbb{C}_k)} \mathcal{B}. \end{array}$$

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We write the covering relation for this order as $\prec$; thus $s \prec t$ iff $s \sqsubseteq t$, $s \neq t$, and for all u, $s \sqsubseteq u \sqsubseteq t$ implies $u = s$ or $u = t$.

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By instantiating $\mathrm{W}_{\mathcal{A},\mathcal{B}}$ appropriately, we obtain the equivalences corresponding to the EF, pebbling and bisimulation games.

For example, $\mathrm{W}^{\mathbb{E}_k}_{\mathcal{A},\mathcal{B}}$ is the set of all (s, t) which define a partial isomorphism.

Characterization by coKleisli morphisms

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We define $\mathcal{S}(\mathcal{A}, \mathcal{B})$ to be the set of all functions $f : \mathbb{C}_k A \rightarrow B$ such that, for all $s \in \mathbb{C}_k A$, $(s, f^*(s)) \in \mathsf{W}_{\mathcal{A},\mathcal{B}}$.

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A *locally invertible pair* (F, G) from $\mathcal{A}$ to $\mathcal{B}$ is a pair of sets $F \subseteq \mathcal{S}(\mathcal{A}, \mathcal{B})$, $G \subseteq \mathcal{S}(\mathcal{B}, \mathcal{A})$, satisfying the following conditions:

1. For all $f \in F$, $s \in \mathbb{C}_k A$, for some $g \in G$, $g^* f^*(s) = s$.
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Proposition

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If $\mathcal{A}$ and $\mathcal{B}$ are finite, so is S, and we can construct the greatest fixpoint by a finite descending sequence $S \supseteq \Theta(S) \supseteq \Theta^2(S) \supseteq \cdots$.

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$$\Delta(G) = \{f \in S \mid \forall s \in \mathbb{C}_k A . \exists g \in G. \, g^* f^* s = s\}.$$

These functions are monotone. Moreover, a pair of sets (F, G) is locally invertible iff $F \subseteq \Delta(G)$ and $G \subseteq \Gamma(F)$.

Thus existence of a locally invertible pair is equivalent to the existence of non-empty F such that $F \subseteq \Theta(F)$, where $\Theta = \Delta\Gamma$.

Since Θ is monotone, by Knaster-Tarski this is equivalent to the greatest fixpoint of Θ being non-empty. (Note that $\Theta(\varnothing) = \varnothing$).

If $\mathcal{A}$ and $\mathcal{B}$ are finite, so is S, and we can construct the greatest fixpoint by a finite descending sequence $S \supseteq \Theta(S) \supseteq \Theta^2(S) \supseteq \cdots$.

This fixpoint is non-empty iff $\mathcal{A} \leftrightarrow^{\mathbb{E}}_k \mathcal{B}$.

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We can now complete our results on logical equivalences.

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Theorem

For structures $\mathcal{A}$ and $\mathcal{B}$:

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(1) & \mathcal{A} \equiv^{\exists\mathcal{L}_k} \mathcal{B} & \Longleftrightarrow & \mathcal{A} \rightleftarrows_k \mathcal{B}. \\
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(3) & \mathcal{A} \equiv^{\mathcal{L}_k(\#)} \mathcal{B} & \Longleftrightarrow & \mathcal{A} \cong_{\mathsf{Kl}(\mathbb{C}_k)} \mathcal{B}.
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Thus in each case, we capture the salient logical equivalences in syntax-free, categorical form.

Coalgebraic characterization of combinatorial parameters

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- *For the pebbling comonad, the coalgebra number of $\mathcal{A}$ corresponds precisely to the* **tree-width** *of $\mathcal{A}$.*
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We thus obtain categorical characterizations of these key combinatorial parameters.

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- Wider horizons: can we connect with significant meta-algorithms, such as decision procedures for guarded logics based on the tree model property, or algorithmic metatheorems such as Courcelle's theorem?
- The wider issue: can we get Structure and Power to work with each other to address genuinely deep questions?